

# Does it Matter whether a Miracle-Like Event Happens to Oneself rather than to Someone Else?

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## *1. An Ancient Controversy*

When Jesus' disciples tell Thomas that they had seen the risen Lord, he notoriously answers: "Unless I see the nail marks in his hands and put my finger where the nails were, and put my hand into his side, I will not believe it." (John 20: 25) When Jesus then appears to him in person, Jesus admonishes him: "Because you have seen me, you have believed; blessed are those who have not seen and yet have believed." (John 20: 29)

This biblical passage raises an interesting puzzle. Should one attach the same evidential weight to miracle-like events when they happen to other people as when they happen to oneself? Does testimony have the same evidential weight as personal experience of miracle-like events? It is not untypical for people to have a religious conversion when they themselves are touched by the hand of God in some way or other, while they attach little weight to testimony from others of events that seem at least equally miraculous. Is this a reasonable attitude?

There is one track of response that goes back to Hume's famous essay 'Of Miracles' (1748) and was recently discussed in Earman (2000). The core idea is that third-person reports, and in particular a chain of third-person reports, tend to be less reliable than first-person experiences. If this is the case, then it is clearly reasonable to attach more weight to personal experience than to testimony. Thomas's response fits in this category—he does not consider the reports of the other disciples to be reliable and Jesus admonishes him for his lack of trust.

But now suppose that, in some particular case, a third-person report of a miracle-like event is equally reliable as one's first-person experience. After all, there are sources that we can trust and also our own senses can be subject to delusions. What are we to say then? Might it still be reasonable to attach more weight to our personal experience than to testimony?

William James and William Alston are at loggerheads with respect to this question. They discuss the evidential value of *mystical experiences*, but at least judging from Alston's examples, these mystical experiences are very much like our miracle-like events. For James, testimony has less evidential value than personal experience. He

writes that “(1) mystical states, when well developed, usually are, and have the right to be, absolutely authoritative over the individuals to whom they come” and “(2) [n]o authority emanates from them which should make it a duty for those who stand outside of them to accept revelations uncritically.” (1902, p. 14) William Alston (1991, p. 280) spells out the following interpretation of James’s position: a religious belief  $P$  is justified when formed directly on the basis of personal mystical experience, but it is not justified when formed on the basis of testimony by a person who came to be justified in believing  $P$  on the basis of personal mystical experience. Alston objects to this position. He submits that if  $X$  is justified in believing some proposition  $P$  on grounds of a personal experience, then  $Y$  is no less justified in believing  $P$  on grounds of  $X$ ’s testimony, provided that  $Y$  is justified in believing that  $X$  is competent, reliable and authoritative. This principle, according to Alston, is acceptable for empirical beliefs and to reject it for religious belief is to hold up a double standard. Mystical experiences are notoriously less reliable than ordinary sensory experiences, but this problem is equally prevalent for personal experience as for experiences that we learn about through witness reports. He writes: “[I]f those not blessed with first hand experience of God cannot become justified in their belief about God from testimony of those who are so blessed, then we are of all men the most miserable.” (1991, pp. 282-93)

## 2. *The Naïve Argument*

Let us first see what the argument might be on Alston’s side. We could proffer the following simple Bayesian argument. Let a miracle-like event be an event that is seemingly indicative of the existence of an all-good, all-knowing and all-powerful being. Such an event might occur in a naturalistic world but this would be very improbable. Let  $E(i)$  be the proposition that such an event happened to a particular person  $i$ . For simplicity, assume that the experience as well as the report of such a miracle-like event are both veridical—i.e.  $E(i)$  is true. Let  $G$  be the proposition that there is a God. Then for any particular person  $i$  the following holds:

$$(1) \quad 1 > P(E(i)|G) = r \gg P(E(i)|\neg G) = x \approx 0 \text{ but } x \neq 0.$$

Let me explain: if there is a God, then there is chance  $r$  (though not a certainty) that he will reveal himself to a particular person  $i$  during her life-time through some miracle-like event. If there is no God, then the occurrence of such an event in this person’s life-time is close to 0, yet it is not impossible in a naturalistic world. Let my prior odds for the existence of God be  $O(G) = P(G)/P(\neg G)$ . For simplicity, let us set these odds at 1—I find it equally likely that there is or that there is not a God. Then my posterior odds that there is a God, after learning that a miracle-like event happened to me or that a miracle-like event happened to, say, John Smith is simply:

$$(2) \quad O(G | E(i)) = \frac{P(E(i) | G)}{P(E(i) | \neg G)} O(G) = r/x$$

So my posterior odds that there is a God are magnified by factor  $r/x$ , whether the miracle-like event happened to  $i = \text{me}$  or to  $i = \text{John Smith}$ .

### 3. Protocols

My argument in this section builds on Bovens and Leeds (2002) and Bovens and Ferreira (2010). The simple Bayesian argument in the previous section commits a standard mistake in Bayesian updating, which is discussed in Shafer (1985). When we are informed of some proposition by a reliable informant, we do not only learn the proposition in question, but we also learn that this item of information was provided to us as one of the items of information that might have been provided to us in the context in question. The *protocol* is the information-generating process. It determines the various items of information that may be provided to us and our credence of receiving each such item of information given various states of the world. We should then update not on the proposition in question, but rather on the fact that we learned that it was this item of information that was provided to us rather than any other that could have been provided to us in the context in question.

This point is particularly salient in the Three Prisoner Problem. Suppose that you, prisoner  $a$ , are one of three prisoners  $a$ ,  $b$  and  $c$ . You know that exactly one of the three prisoners will be executed and you assign an equal *ex ante* credence to each prisoner being executed. Let  $A$  be the proposition that  $a$  will be executed,  $B$  the proposition that  $b$  will be executed, and  $C$  the proposition that  $c$  will be executed. Then  $P(A) = 1/3$ . There is a guard whom you know will answer your question in a truthful manner. You ask her to name exactly one of the other two prisoners who will *not* be executed. Clearly, she can always name one such prisoner. If it is  $b$  who will be executed, she will name  $c$ , if it is  $c$ , she will name  $b$  and if it is  $a$  (i.e. you) who will be executed, then she will flip a coin to determine whether to name either  $b$  or  $c$ . You know all this. The guard answers “ $b$  will not be executed”.

It would be erroneous to reason as follows. You learned the proposition  $\neg B$ , i.e. that  $B$  will not be executed. So you update on the proposition that you learned, *viz.*  $\neg B$ :

$$(3) \quad P(A \mid \neg B) = \frac{P(\neg B \mid A)P(A)}{P(\neg B)} = \frac{1 \times 1/3}{2/3} = 1/2$$

This answer is incorrect—clearly you should not change your credence that you will be executed from  $1/3$  to  $1/2$ . The information that the guard provides you should not affect your *ex ante* credence that you will be executed.

What is the correct way of reasoning about this problem then? What you learn from the guard is not just the proposition that  $b$  will not be executed. You learn more. You learn that the item of information “ $b$  will not be executed” has been provided to you (by a truthful guard) as one of the two items of information that might have been provided to you, *viz.* “ $b$  will not be executed” and “ $c$  will not be executed” in the context in question. We construct the conditional probability table in which  $INF$  is the variable that will take on “ $\neg B$ ” if the item of information that is provided to you is “ $b$  will not be executed” (etc) and @ is the variable that will take on the value  $A$  if  $A$  is true in the actual world (etc) in Table 1.

<i>Three Prisoners</i>		@ =		
		A	B	C
$P(INF   @)$				
$INF =$	“¬B”	½	0	1
	“¬C”	½	1	0

Table 1. The Protocol in the Three Prisoners Problem

We now update on “¬B”, i.e. that the guard has provided you with the information “*b* will not be executed”:

$$\begin{aligned}
 (4) \quad P(@ = A | INF = \text{“}\neg B\text{”}) &= \frac{P(INF = \text{“}\neg B\text{”} | @ = A)P(@ = A)}{P(INF = \text{“}\neg B\text{”})} \\
 &= \frac{P(INF = \text{“}\neg B\text{”} | @ = A)P(@ = A)}{\sum_{@=A,B,C} P(INF = \text{“}\neg B\text{”} | @)} \\
 &= \frac{1/2 \times 1/3}{1/3(1/2 + 0 + 1)} = 1/3
 \end{aligned}$$

This is obviously the correct result. The answer of the guard should not affect your *ex ante* credence that you will be executed. (Pearl, 2000: 58-9 with reference to Gardner, 1961)

So how does this insight apply to our problem?

One could ask: What is the typical protocol that leads someone to learn that a miracle-like event happened to herself? What is the typical protocol that leads someone to learn that a miracle-like event happened to John Smith (i.e. a particular person other than herself)?

Let me introduce some terminology. Let a *holy person* be a person to whom some miracle-like event has happened. Let a *prophet* be a person who knows who is and who is not a holy person within a particular population.

Now people who learn about John Smith being a holy person typically do so in a particular way. They tend to consult prophets and ask them about whether there is anyone who has been the subject of miracle-like events. Furthermore, let us suppose that, within the context in question, a person in search of religious truth consults a single prophet who has intimate knowledge of the lives of *n* people so that she knows whether miracle-like events happened or did not happen in each person’s life. Either the prophet will say “(A miracle-like event happened to) None” or she will provide a single name, e.g. “John Smith”. If she provides a single name, then this does not

mean that a miracle-like event only happened to exactly one person. It could have happened to multiple people, but, for the sake of simplicity, let us suppose that the prophet will just provide exactly one name by randomising under a uniform distribution over the persons to whom a miracle-like event happened. So suppose that there are two people of whose lives the prophet has intimate knowledge, viz. John Smith and Mary Smith. Then the prophet will answer one of the set {"None", "JS", "MS"}. Let us call this protocol '*Beatist*' since it describes a person who, in her search for religious truth, takes an interest in whether there do or do not exist holy persons – i.e. *beati* – in this world.

What about conversions on grounds of miracle-like events happening to oneself? What is the typical way in which one comes to learn about a miracle-like events happening to oneself? What comes to mind is the story of St. Paul. St. Paul persecuted early Christians and he only converts after being blinded and receiving a vision of the resurrected Christ. Now St. Paul did not knock on a prophet's door to learn about holy people—he could not care less about the existence of holy persons before his conversion. Rather, through life's lessons, St. Paul would either be confronted by a miracle-like event or not. His own life would provide one answer out of the set {"Miracle-like event happening to St. Paul", "No miracle-like event happening to St. Paul"} or, for short, {"SP", "¬SP"}. Let us call this protocol '*St. Paul*'.

In Tables 2 and 3, I construct conditional probability tables for these protocols in which *INF* is the random variable that can take on the various items of information that may be in my data base, given my epistemic interests and @ is the random variable that take on the value G, if God exists or ¬G if God does not exist.

Table 2 may require some clarification. The chance of no miracle-like event happening to either JS or MS conditional on God's existence is  $(1-r)^2$  (on the assumption that miracle-like events happening to JS and MS are independent conditional on God's existence or non-existence). Similarly, the chance of no miracle-like event happening to either JS or MS conditional on God's non-existence is  $(1-x)^2$ . Furthermore, the chance of a miracle-like event happening to at least one of JS or MS conditional on God's existence is  $(1-(1-r)^2)$ . Now if it happened to at least one person then there is an equal chance that the prophet will say "JS" or that he will say "MS" and so the chance that the prophet will say "JS" conditional on God's existence is  $\frac{1}{2}(1-(1-r)^2)$ . Or here is another way to see this. The chance that the prophet will say "JS" equals the chance that it happened to JS and not to MS (i.e.  $r(1-r)$ ) plus the chance that it happened to both JS and MS and that JS's was drawn by randomisation (i.e.  $\frac{1}{2} r^2$ ). Now  $r(1-r) + \frac{1}{2} r^2 = \frac{1}{2}(1-(1-r)^2) = r - \frac{1}{2} r^2$ . And similarly for the other entries in Table 2.

<b>Beatist</b>		@ =	
$P(INF   @)$		G	$\neg G$
$INF =$	“None”	$(1-r)^2$	$(1-x)^2$
	“JS”	$\frac{1}{2}(1-(1-r)^2)$	$\frac{1}{2}(1-(1-x)^2)$
	“MS”	$\frac{1}{2}(1-(1-r)^2)$	$\frac{1}{2}(1-(1-x)^2)$

Table 2. The *Beatist* Protocol

<b>St. Paul</b>		@ =	
$P(INF   @)$		G	$\neg G$
$INF =$	“SP”	$R$	$x$
	“ $\neg$ SP”	$1-r$	$1-x$

Table 3. The *St. Paul* Protocol

We can now calculate:

$$(Beatist) \quad O_{Beatist}(@ = G | INF = \text{“JS”}) =$$

$$\frac{P(INF = \text{“JS”} | @ = G)}{P(INF = \text{“JS”} | @ = \neg G)} O(@ = G) = \frac{1/2(1-(1-r)^2)}{1/2(1-(1-x)^2)}$$

$$(St. Paul) \quad O_{St. Paul}(@ = G | INF = \text{“SP”}) =$$

$$\frac{P(INF = \text{“SP”} | @ = G)}{P(INF = \text{“SP”} | @ = \neg G)} O(@ = G) = r/x$$

We can generalise *Beatist* for a prophet who has intimate knowledge of the lives of  $n$  people:

$$(Beatist) \quad O_{Beatist}(@ = G | INF = \text{“JS”}) = \frac{1/n(1-(1-r)^n)}{1/n(1-(1-x)^n)}$$

To see that under the constraints in (1) and for  $n > 1$ ,  $O_{St. Paul}(@ = G | INF = \text{"SP"}) > O_{Beatist}(@ = G | INF = \text{"JS"})$ , a proof in the appendix of the following fact is included:

$$(5) \quad \frac{O_{StPaul}(@ = G | INF = \text{"SP"})}{O_{Beatist}(@ = G | INF = \text{"JS"})} = \frac{r (1 - (1 - x)^n)}{x (1 - (1 - r)^n)} > 1$$

To get a feel for the numbers, let us set  $r = .50$ ,  $x = .01$ . Then the odds for *St. Paul* are 50 and for *Beatist* are a function of  $n$ , as we can see in Figure 1. For  $n = 5$ , the odds are 19.77.

[Figure 1 about here.]

As  $n$  approaches infinity, the limit for the odds is 1 (i.e. the prior odds) for *Beatist*. This is easy enough to explain. If there is a large number of people to whom miracle-like events may happen, then the evidentiary value decreases as there is bound to be someone to whom something improbable happened, even in the absence of there being a self-revealing God.

It is also clear from the equations (*Beatist*) and (*St. Paul*) that the odds are increasing in  $r$  – the more God is a revealing God, the more a miracle-like event is evidence for his existence, *ceteris paribus* – and decreasing in  $x$  – the more improbable the miracle-like event is in a naturalistic world, the more its occurrence is evidence for God’s existence, *ceteris paribus*. Furthermore, as  $x$  approaches 0, the posterior odds go to infinity.

#### 4. Discussion

Let us assume that we do not question the actual occurrence of the miracle-like event and that what is at stake is the rationality of religious belief on grounds of the veridical experience of or report about miracle-like events. Now of course miracle-like events come in various shapes—they range all the way from Jesus rising from the dead to a synagogue remaining standing in an earthquake. Clearly  $x$  is much closer to 0 for the former than for the latter. We have seen that, as  $x$  approaches 0, the posterior odds go to infinity in the limit for both the *St. Paul* and the *Beatist* protocols. So the interesting cases are the cases in which  $x$  is low but we are in the realm of uncanny events that seem to have the mark of an all-good, all-knowing and all-powerful being.

I also bracket the case in which a person claims that she *feels* that God “spoke to her” through some miracle-like event happening to herself. Whether one feels called upon by God in experiencing a miracle-like event, is not a function of its evidentiary value. To speak with Pascal, such matters may rest on reasons of the heart that the intellect does not understand.

My argument comes in when someone wishes to make her faith contingent on the evidentiary value of a miracle-like event that is an uncanny event and that seems to have God’s name written on it. In this case, the protocol of how we come to learn

about such miracle-like events does matter a great deal. I have spelled out *typical* protocols of how one comes to learn a proposition that a miracle-like event happened to another person and of how one comes to learn a proposition that a miracle-like event happened to oneself. And this difference in protocol matters a great deal to the evidentiary value of the proposition. But how typical is *typical*?

I claimed that *St. Paul* was a *typical* protocol for someone who lets her religious beliefs be affected by miracle-like events happening to herself. But does it need to be the case that, if I change my religious beliefs on grounds of a miracle-like event happening to myself, then I was only open to miracle-like events happening to myself? Not really. We *could* imagine a St. Paul who is just as open to a prophet reporting on the existence of a holy person amongst the people of whose lives he has intimate knowledge. However, the prophet remained silent before the miracle-like event happened to St. Paul and the prophet's confirmation of the miracle-like event that happened to St. Paul is the only evidence St. Paul has to go on. Then we would need to analyse the person who experienced a miracle-like event herself based on the *Beatist* protocol and not on the *St. Paul* protocol. But when someone converts on the basis of miracle-like events happening to herself, she is typically portrayed as a person who is not consulting prophets, who is not open to such events happening to others. She is *a St. Paul*, i.e. the kind of person whose religious beliefs can only be affected by miracle-like events happening to herself.

I claimed that *Beatist* was a *typical* protocol for someone who lets her religious beliefs be affected by miracle-like events happening to other people. But does it need to be the case that if I change my religious beliefs on grounds of a miracle-like event happening to someone else, then I was open to the reports of a prophet reporting on a holy person? Again, not really. It might well be the case that I am completely tuned into exactly one person, other than myself, in this world and only miracle-like events happening to that person could affect my religious faith. Then we would need to analyse the person who changes his religious beliefs on grounds of a miracle-like event happening to that person as a St. Paul and not as a Beatist. But when someone converts on the basis of miracle-like event happening to other people, she typically is open to prophets reporting on the existence of some holy person in a population of people.

Why did we constrain the prophet in the *Beatist* protocol to naming the name of *one* holy person? Might a real-life Beatist not wish to hear about other people to whom miracle-like events have happened? Sure—and we could spell out a protocol for the possible answers of the prophet which includes multiple holy persons. But note that, in the case at hand, we are updating on the information “John Smith” and not on the information “John Smith and nobody else”. So we would need to stipulate some decision procedure for the prophet which is such that the prophet saying “John Smith” is not tantamount to a miracle-like event happening to John Smith and to nobody else. And this would require additional stipulations. We chose for the simplest model, i.e. to just let the prophet name exactly one holy person. And not only is this the simplest model, there is also some basis in Beatist psychology for the assumption of naming a single person. Many Beatists are satisfied when they hear of the existence of one holy person. Or at least they do not try to determine whether there are holy persons outside their faith. Once they have determined that Jesus is a holy person, they do not care about finding out whether other faiths rest on reliable holy-person accounts as well.

I have constructed a dichotomy of protocols. But of course in the real world these protocols are only the extreme points on a continuum. Furthermore the prophet is just a personification of the societal lore of reliable reports on miracle-like events. And this societal lore may afford a more or less direct connection between the people who experienced a miracle-like event and the prospective believers. With reliable twitter reports, the connection is more direct, whereas with reliable reports through the standard media, the connection is less direct. Some people are more open to this lore than others. For example, some may be open to miracle-like events happening not only to themselves but also to close acquaintances. That places them somewhere in between a St. Paul and a Beatist. I have provided an analysis of the extreme points on this continuum and we simply need to interpolate for the intermediate cases.

As an aside, let me remark that my argument is also relevant to miracle-like events happening to followers of cult-figures whose beatification and sainthood is under consideration. José Manuel Almuzara is leading a campaign to beatify Antoni Gaudí. The occurrence of an uncontested miracle-like event would help the cause. Almazura hopes that with the Pope Benedict XVI's visit to Barcelona the number of people praying to Gaudí will go up and along with it the chances of a miracle-like event: "It is not the same if only 50 people are praying to him for help as when five million are praying," he is quoted to be saying. (Tremlett, 2010) In this case the difference is not whether we are open to reports from 50 or five million people, but rather the difference is between whether the desired miracle-like events might happen to 50 or to five million people. Now it may well be that the chances of a miracle-like event happening to at least one of Gaudí's followers may go up as the numbers increase. But my argument applies to this case as well: One should not forget that, with increasing numbers of followers, the evidential value of such a miracle-like event would go down.

Let us return to the central lesson, viz. if a person wants to change her religious beliefs on the basis of the evidentiary value of miracle-like events (where miracle-like events are to be understood as low-probability events that seem to bear the hand of an all-good, all-knowing and all-powerful being), then it does make a difference whether she goes through the world with an openness to such events happening to herself or whether she goes through the world with an openness to reports from reliable sources who provide access to a large database of potential subjects of miracle-like events. There are miracle-like events which are such that, if they happen to someone with a St. Paul-like attitude, then this should have great evidential value for her, but if someone with a Beatist-like attitude were told by a fully reliable judge that they happened to, say, a 13<sup>th</sup> century monk, then this should have little evidential value for her.

There remains the question of what the lesson is for *my own* belief updating? What am *I* entitled to believe on grounds of miracle-like events happening to others as opposed to myself? After all, I am just one person with one attitude. Certainly—but I am also not completely transparent to myself and I can learn about what kind of person I am from the fact that I find myself pondering one thing or another. If I am pondering what I should believe on grounds of a miracle-like event that happened to a 13<sup>th</sup> century monk and that has been told to me by a reliable source, then I am probably more of a Beatist than a St. Paul. And if I am pondering what I should

believe on grounds of a miracle-like event that has happened to me, then I am probably more of a St. Paul than a Beatist. And as we have seen, miracle-like events have more evidential value to a St. Paul than to a Beatist. So it is perfectly reasonable to let miracle-like events that happen to myself have more evidential value than miracle-like events that happen to someone else. And this vindicates James against Alston.<sup>1</sup>

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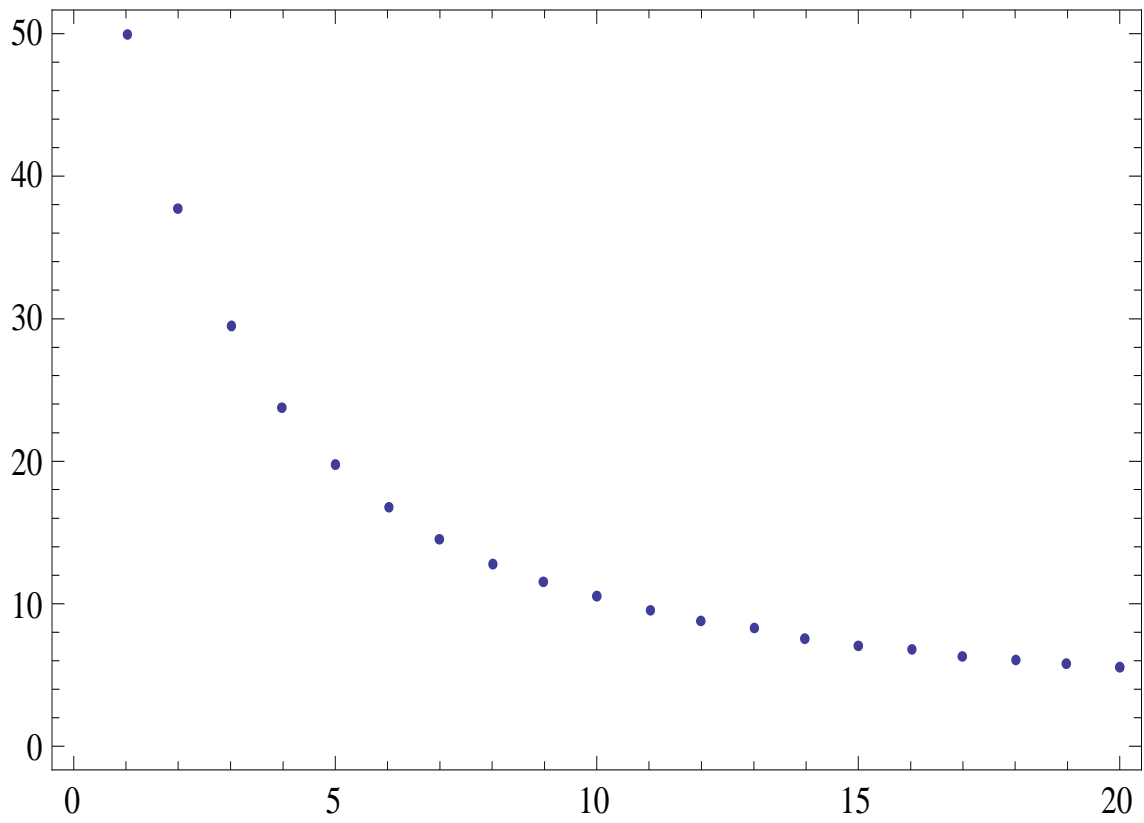
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*n*

Figure 1. The posterior odds of the God-hypothesis in *Beatist* as a function of  $n$  with  $n$  being the number of people of whom the prophet has intimate knowledge for  $r = .5$  and  $x = .01$

**Appendix** (by Claus Beisbart)

*Theorem.* If  $r$  and  $x$  are real numbers with  $1 > r > x > 0$  and  $n$  is an integer greater than 1, then  $\frac{r(1-(1-x)^n)}{x(1-(1-r)^n)} > 1$ .

*Proof.* Since  $1 > r > 0$  and thus  $1 - (1-r)^n > 0$ ,

$$(1) \quad \frac{r(1-(1-x)^n)}{x(1-(1-r)^n)} > 1$$

holds if and only if

$$(2) \quad \frac{1-(1-x)^n}{x} > \frac{1-(1-r)^n}{r}$$

Define  $f(z) = \frac{1-(1-z)^n}{z}$ . So we need to show that  $f(r) > f(x)$  for  $1 > r > x > 0$ . This is so if  $f(z)$  is a strictly monotonically decreasing function in the range  $(0,1)$ , or, in other words, if  $f'(z) < 0$  for  $z \in (0,1)$ . We calculate:

$$(3) \quad f'(z) = \frac{nz(1-z)^{n-1} - (1-(1-z)^n)}{z^2}$$

The denominator in (3) is clearly positive, so we need to show that the numerator is negative for  $z \in (0,1)$ . Note the following equality:

$$(4) \quad \begin{aligned} nz(1-z)^{n-1} - (1-(1-z)^n) &= (nz + (1-z))(1-z)^{n-1} - 1 \\ &= ((n-1)z + 1)(1-z)^{n-1} - 1 \end{aligned}$$

Hence we need to show that

$$(5) \quad ((n-1)z + 1)(1-z)^{n-1} < 1$$

for  $n$  is an integer greater than 1. We show this by means of mathematical induction. Clearly for  $n = 2$ , (5) holds since  $(z+1)(1-z) = 1 - z^2 < 1$ , since  $z^2 \in (0,1)$  for  $z \in (0,1)$ . Now suppose that (5) holds for some integer  $n > 1$ . Then we will show that it also holds for  $n + 1$ . So we need to show that

$$(6) \quad ((n+1-1)z + 1)(1-z)^{n+1-1} < 1$$

Note the following equality:

$$(7.1) \quad ((n+1-1)z + 1)(1-z)^{n+1-1}$$

$$(7.2) \quad = ((n-1)z + z + 1)(1-z)^{n-1}(1-z)$$

$$(7.3) \quad = ((n-1)z + 1)(1-z)^{n-1}(1-z) + z(1-z)^{n-1}(1-z)$$

$$(7.4) \quad = ((n-1)z + 1)(1-z)^{n-1}(1-z) + z(1-z)^n$$

From (5), we know that the first addend in (7.4) is smaller than  $(1-z)$ . Hence, the expression in (7) is smaller than  $(1-z) + z(1-z)^n$ . We also know that  $(1-z)^n < 1$  for  $z \in (0,1)$ . Hence,  $z(1-z)^n < z$ , and so  $(1-z) + z(1-z)^n < (1-z) + z = 1$ . It follows that the expression in (7) is indeed smaller than 1 and so that the induction step in (6) holds. This concludes the proof.