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A power measure analysis of Amendment 36 in Colorado

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Abstract Colorado's Amendment 36 proposed to switch Colorado's representation in the Electoral College from winner-takes-all to proportionality. We evaluate unilateral and uniform switches to proportionality both from Colorado's perspective and from an impartial perspective on the basis of a priori and a posteriori voting power measures. The present system is to be preferred to a unilateral switch from any perspective on any measure. A uniform switch is to be preferred to the present system from Colorado's perspective on an a priori measure, and from an impartial perspective on an a posteriori measure. The present system is to be preferred to a uniform switch from Colorado's perspective on an a posteriori measure (with some qualifications), and from an impartial perspective on an a priori measure. We conclude with a discussion of the appropriateness of these measures.

Keywords US presidential elections · Colorado's Amendment 36 · A priori and a posteriori measures of voting power · Mean majority deficit

1 Introduction

On November 2, 2004 the citizens of Colorado not only cast their vote for the President of the United States. They also decided how Colorado will be represented in the Electoral College (EC). As most of the other states, Colorado has a winner-takes-all system (A), i.e. all electoral votes from Colorado are given to the candidate with the most votes within Colorado. Amendment 36 proposed to change this to proportional representation (P).[1] If it

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[1] See http://www.lawanddemocracy.org/pdffiles/COamend36.pdf for Amendment 36.

had been accepted, the nine electoral votes of Colorado would have been divided between the different candidates in proportion to the votes they receive from the Coloradans.[2]

The amendment was defeated, but it is interesting to see whether this amendment might have been beneficial in some respects. Furthermore it is worth investigating what would happen if all states were to adopt proportional representation. Hence, we will compare three different *voting schemes*:

(A^{51}) Each US state adopts the winner-takes-all system (which is a fair approximation of the present system).

($A^{50}P_C$) Colorado adopts proportional representation, but *all other* US states keep the winner-takes-all system (which would have been a fair approximation, if Amendment 36 had been accepted).

(P^{51}) All US states adopt proportional representation.

In assessing these voting schemes, we will concentrate on the influence of a single citizen's vote as measured by her voting power. Voting powers will be calculated on the base of probabilistic assumptions.

We will examine the issue from two different *perspectives*. Each perspective leads to particular desiderata. From the (self-interested) *perspective of Coloradans*, it would be beneficial to maximize the influence of a Coloradan. From an *impartial perspective*, it would be beneficial to minimize the spread of the influences of citizens from different states and to secure that the election outcomes reflect the preferences of the voters as much as possible. Whereas the perspective of the Coloradans is prudential, the impartial perspective is of a moral nature.

We will consider *two probabilistic models* (cf. Laruelle and Valenciano 2005 for the general framework). Our first model (B) is a priori and leads to the Banzhaf indices of voting power. Our second model (R) is a posteriori and builds upon data from past elections. The choice of a probabilistic model is controversial in the literature (see Felsenthal and Machover 2000, p. 13 and Gelman et al. 2004).

We will combine each model with each perspective. This yields four comparisons of the alternative voting schemes. The comparisons can be organized in a (2×2)-matrix. We consider both perspectives under each of the models for two reasons. First, it is not entirely clear what model is more appropriate for each perspective. Second, we consider a comparative application of both models as illuminating in its own right.

The US presidential elections and the EC have been of enduring interest in political discussion and academic research for decades (see Grofman and Feld 2005 for a recent discussion of objections against the EC). Much of the controversy focuses on the question whether the system is fair, particularly as far as the relative influence of citizens from smaller and larger states is concerned. In the literature, several measures can be found for approaching the fairness issue. Banzhaf III (1968) applies Banzhaf voting power to the EC (see Felsenthal and Machover (1998, Chap. 3) for an introduction to Banzhaf voting power). Rabinowitz and MacDonald (1986) investigate the EC by using alternative measures of power based on a two-dimensional policy space. Brams and Davis (1974) (see also Colatoni et al. 1975 and Bartels 1985) study the resources that candidates have or should have allocated to the different states.

We share a concern about fairness in our evaluation of the EC and apply similar techniques and models. Particularly, we will quantify power in terms of Banzhaf voting power.

[2]Note that the winner-takes-all system has not always been predominant in the past, see (Grofman and Feld 2005, footnote 1).

But unlike much other research in this field, we will not take issue with the composition of the EC. Instead we treat its present composition as given and study how modifications to the determination of the electors in the EC affect the voting system. Because our research is prompted by Amendment 36, we also stress the perspective of citizens from a particular state, viz. Colorado.

The plan of our paper is as follows. We provide the theoretical framework for our discussion and describe the models in Sect. 2. In Sect. 3 we calculate the distributions of voting power for the different voting schemes. We provide a comparison of the alternative voting schemes from both perspectives and under both models in Sect. 4.

2 Voting power in the US presidential elections

For our analysis it is useful to model the presidential elections under the alternative voting schemes as composite voting games $\mathcal{W}$ (Felsenthal and Machover 1998, Definition 2.3.12, p. 27). To do so we adopt the following assumptions for every voting scheme:

1. There are only two candidates X and Y. We can therefore conceive of the election as a vote on whether X will be president. This is an idealization, because there are often third candidates.
2. The winner of the presidential elections is uniquely determined by the votes of the people. This is an idealization, because draws are possible in the Electoral College (269 vs. 269 votes), in which case the decision goes to the Congress. We neglect the Congress and model the presidential elections as an equiprobable lottery over two voting games instead of a single voting game (Laruelle and Valenciano 2004). In the first (second) voting game X (Y) is declared winner in case there is a draw in the EC. We obtain voting powers by averaging over the voting powers in both voting games. For the sake of brevity, we will suppress the talk of lotteries in the following.
3. All states except Colorado adopt the winner-takes-all system. This is an idealization, because in Maine and Nebraska different systems are implemented, which allow for split votes by their representatives.

We will evaluate the different voting schemes on the basis of voting power. The voting power of a citizen $c^{(i)}$ from state i in a two-tier voting system is the probability that she is doubly pivotal, $P(c^{(i)} \to us)$ (cf. Felsenthal and Machover 1998, Definition 3.2.2, p. 39). A citizen is doubly pivotal, if and only if, had she voted differently, the outcome of the presidential elections would have been different. For this to be the case (a) she must be pivotal in her own state ($c^{(i)} \to i$), i.e. her state would have sent a different set of electors, had she voted differently; and (b) her state must be pivotal, i.e. had her state sent this different set of electors, a different candidate would have been elected ($i \to us$).

For our first model (B), we adopt the assumptions of Banzhaf voting power. That is, the votes of individual citizens and therefore also of the states are stochastically independent, and every voter votes in favor of X with probability 0.5.

Our second, more realistic, model (R) is adapted from Crain et al. (1993). It can be described in terms of sentiment indices, where sentiment indices quantify the proneness to vote for candidate X. A national vote can be simulated as follows:

1. A national sentiment index μ_{us} is randomly drawn.
2. For each state i state-wise sentiment indices μ_is are determined by

$$\mu_i = a_i \mu_{us} + b_i + \delta_i . \tag{1}$$

3. Given some value of μ_i, the citizens in state i are independently assigned votes, where every voter in state i votes for X with a probability of μ_i.

Under this model, the national sentiment index μ_{us} takes account of the fact that certain events (political, economic, ...) will affect the decisions of all voters. In having state-specific biases a_i and b_i we assume that citizens from different states react differently to such events. δ_i captures the impact of state-specific events. Since all citizens' votes are determined on the basis of the national sentiment index, the model displays *inter*state and *intra*state correlations amongst the votes.[3]

We assume normal probability distributions for the random variables μ_{us} and δ_i whose means and variances are determined by fitting the model to data from the ten past elections (1968–2004).[4]

In order to be consistent with our assumption of two candidates, we have to abstract from votes that were cast for third candidates. Therefore, for each election in our data set, the votes for third candidates are given to the Democrat and the Republican candidates in proportion to the votes they received in this election.

For each election α in our data set we order the states i according to their fractions f_i^α of votes for the Democrat candidate. The national sentiment index for this election μ_{us}^α is taken to be the fraction $f_{i_0}^\alpha$ in the median state i_0. For each state i we obtain values for a_i and b_i by doing a linear regression of the f_i^αs against the μ_{us}^αs. The variance for δ_i is obtained from the residuals of the best fit using standard methods.[5] The mean of δ_i is set to zero. The normal distribution over μ_{us} has the mean and the variance of the μ_{us}^αs.

In order to measure the spread of the voting powers of citizens from different states, we use the standard deviation of all citizens' voting powers as well as an inequality index J^{DP} recently introduced by Laruelle and Valenciano (2004, (7)) (see also Laruelle and Valenciano 2002). This index is uniquely characterized by a set of desirable requirements.

In order to measure in how far the outcome reflects the preferences of the majority of voters we use the measure Δ, i.e. the Mean Majority Deficit (MMD). The majority deficit for a particular profile is set at zero, if the majority of the votes agrees with the outcome of the election. If it does not agree, then the majority deficit equals the size of the majority minus the size of the minority. The MMD is the expectation value of the majority deficit under some probability distribution over the profiles (cf. Felsenthal and Machover 1998, Definition 16, p. 60). It can be straightforwardly determined by means of simulations. Under model B, there is an analytic relation between the MMD and voting powers (see Felsenthal and Machover 1998, Theorem 3.3.17, p. 60, and Approximation 3.3.10, p. 56).[6]

3 Results: the voting power distribution for the alternative voting schemes

We label the US states and Washington DC by numbers $i = 1, \ldots, 51$, where 51 denotes Colorado. Let N be the number of votes in the US, and N_i be the number of votes in state i.

[3]Note that the model on which the Banzhaf indices are based can be regained from (1). For instance, we assume that μ_{us} is constant at 0.5 and that δ_i is constant at 0, whereas a_i and b_i are set at 1 and 0, respectively for all states i.

[4]The data were obtained from http://uselectionatlas.org/RESULTS/index.html.

[5]That is, we take $\sqrt{\chi^2/(N_{el}-2)}$ as an estimate of the standard deviation, see Breiman (1973, p. 315).

[6]Hinich (1975, pp. 7–12) instead consider the probability that the majority of the votes does not agree with the outcome of the election.

We assume that for every state i, the number of votes N_i is the product of the turnout and the voting age population, VAP_i. We assume a turnout of 0.6 for every state.[7]

3.1 Winner takes all for every state (A^{51})

Taking into account assumptions 1–3 the present US system can be modeled as the composite voting game

$$\mathcal{A}^{51} = \mathcal{V}[\mathcal{W}_1, \ldots, \mathcal{W}_{51}]. \tag{2}$$

The $\mathcal{W}_i$s are the simple majority voting games that pick the electors for each US state ($i = 1, \ldots, 51$), and $\mathcal{V}$ is a weighted voting game that models the final voting within the EC (Felsenthal and Machover 1998, p. 29).

Model B Under our first model, the conditions (a) and (b) for double pivotality (Sect. 2) are stochastically independent. Hence,

$$P(c^{(i)} \to us) = P(c^{(i)} \to i) \times P(i \to us). \tag{3}$$

The first factor on the r.h.s., $P(c^{(i)} \to i)$, has been shown to approximate

$$P(c^{(i)} \to i) \approx \sqrt{\frac{2}{\pi N_i}} \tag{4}$$

(see the proof of Theorem 3.4.3 in Felsenthal and Machover (1998, p. 67)). The second factor is calculated using the power measure program provided at Temple University (http://www.math.temple.edu/cow/bpi.html). Additionally, we ran simulations to check the validity of these results, and we found strong agreement between the simulations and the analytical results.

Results can be seen in the left panel in Fig. 1. $P(c^{(i)} \to us)$, only depends on the VAP and the number of electors of her home state. Roughly, the voting power increases as a function of VAP.[8] The black line in Fig. 1 shows the average over the voting powers of all US citizens. Coloradans have about 71.9% of the average power. Call this fraction the relative share of power for a Coloradan, $\tilde{P}(c^{(i)} \to us)$. The relative share of power equals the normalized Banzhaf index divided by $1/N$, where N is again the number of votes in the US (cf. Felsenthal and Machover 1998, Definition 3.3.2 p. 39).

Modell R For calculating the voting power of a citizen $c^{(i)}$ under our more realistic model R we start with conditionalizing on μ_{us}:

$$P(c^{(i)} \to us) = \int \mathrm{d}\mu_{us}\, p(\mu_{us}) P(c^{(i)} \to us|\mu_{us}). \tag{5}$$

Here p is the probability density over μ_{us} and $P(c^{(i)} \to us|\mu_{us})$ denotes the conditional probability that $c^{(i)}$ is doubly pivotal. In our calculations we approximate the integral in (5) by a Riemann sum.

[7] We use voting age populations for the US from July 2003 (http://www.census.gov/Press-Release/www/releases/CB04-36TABLE1.pdf). We round to thousands of people.

[8] Our results are in rough agreement with results for 2000 at http://banzhaf.net/ec2000.html.

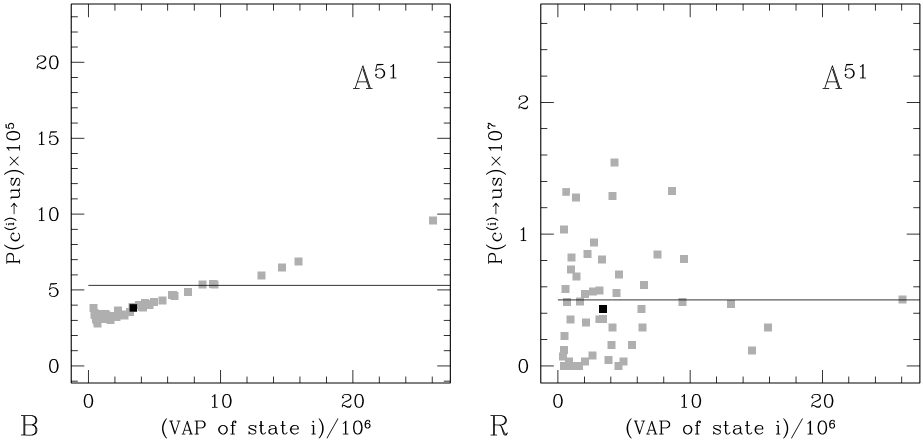

Fig. 1 The probability that a citizen $c^{(i)}$ is doubly pivotal under the present voting scheme A^{51} as a function of VAP of her home state i for $i = 1, \ldots, 51$. *Left panel*: model B. *Right panel*: model R. The *black square* is Colorado. The *black line* is the average of the voting powers from all US citizens

Given a fixed μ_{us}, votes from different states are independent, and therefore $P(c^{(i)} \to us|\mu_{us})$ factorizes into

$$P(c^{(i)} \to us|\mu_{us}) = P(c^{(i)} \to i|\mu_{us}) \times P(i \to us|\mu_{us}). \tag{6}$$

The first factor, $P(c^{(i)} \to i|\mu_{us})$, can be approximated by

$$P(c^{(i)} \to i|\mu_{us}) \approx \frac{p_i(\mu_i = 0.5|\mu_{us})}{N_i} \tag{7}$$

where $p_i(\cdot|\mu_{us})$ is the conditional probability density over μ_i given μ_{us} (Chamberlain and Rothschild 1981, p. 154). Because of (1), $p_i(\cdot|\mu_{us})$ is a normal with a mean of $a_i\mu_{us} + b_i$ and a variance equating the variance of δ_i.

The second factor in (6), the probability that state i is pivotal in the EC, is estimated by simulations. For a fixed μ_{us} we run 1 m realizations. For each realization we randomly draw a δ_i value for each state i and determine the state-wise sentiment indices μ_i according to (1). Using these μ_i-values, the voting profile in the EC is constructed as follows. If $\mu_i \geq 0.5$, the electors from state i vote in favor of X, and they vote for Y otherwise. For each voting profile in the EC we check which states are pivotal. $P(i \to us|\mu_{us})$ is then estimated by the fraction of realizations under which state i is pivotal.[9]

Results can be seen in the right panel of Fig. 1. Under model R, the voting power of a citizen is no longer a function of only VAP and the number of electors. Citizens from states that are close in VAP can greatly differ in terms of voting power. The reason is as follows. Under model R, the chance that a voter in state i is doubly pivotal is also dependent on whether past elections in her state were close when the general election was close.

The relative share of power for a Coloradan is 86% under model R. Again, Coloradans are below average.

[9] Strömberg (2005) has recently proposed an analytical approximation for the second factor in (6). We have tested this approximation and found reasonable agreement.

3.2 Amendment 36 ($A^{50}P_C$)

Let us now suppose that Amendment 36 had been accepted. We model the resulting voting system as the voting game $\mathcal{A}^{50}\mathcal{P}_C$:

$$\mathcal{A}^{50}\mathcal{P}_C = \mathcal{V}'\big[\mathcal{W}_1, \ldots, \mathcal{W}_{50}, \mathcal{W}'_{51,1}, \mathcal{W}'_{51,2}, \ldots, \mathcal{W}'_{51,9}\big] \tag{8}$$

where the voting games $\mathcal{W}'_{51,1}, \ldots, \mathcal{W}'_{51,9}$ have the same assembly, viz. Colorado's population, but different thresholds. You can think of these games in the following way. If at least one Coloradan elector is to vote for X in the EC, at least a fraction of $t^1_{51} = 0.5/9$ of the Coloradans have to vote for X. Thus for the first elector the threshold t^1_{51} is relevant, and there is a voting game $\mathcal{W}'_{51,1}$ for the first elector with a threshold of t^1_{51}. If a second Coloradan elector is to vote for X, at least a fraction of $t^2_{51} = 1.5/9$ of the Coloradans have to vote for X, and there is a voting game $\mathcal{W}'_{51,2}$ for the second elector with a threshold of t^2_{51}, and so on. According to the proposal for Amendment 36 the relevant thresholds for the case of two candidates are $t^k_{51} = (k - 0.5)/9$ for $k = 1, \ldots, 9$.[10] $\mathcal{V}'$ is a variant of $\mathcal{V}$, where Colorado's block vote has been replaced by nine single votes.

Let us now turn to the probabilities of pivotality under $A^{50}P_C$.[11] There are nine possibilities how a Coloradan can be doubly pivotal. She can be doubly pivotal through the kth elector for $k = 1, \ldots, 9$ ($c^{(51)} \stackrel{k}{\to} us$) and to be doubly pivotal through elector k, she must be pivotal in $\mathcal{W}'_{51,k}$ ($c^{(51)} \stackrel{k}{\to} 51$) and representative k must be pivotal in the EC ($51 \stackrel{k}{\to} us$). These possibilities are disjoint, so the probability of a Coloradan being doubly pivotal under the new scheme equals

$$P\big(c^{(51)} \to us\big) = \sum_{k=1}^{9} P\big(c^{(51)} \stackrel{k}{\to} us\big) \tag{9}$$

$$= \sum_{k=1}^{9} P\big(c^{(51)} \stackrel{k}{\to} 51\big) \times P\big(51 \stackrel{k}{\to} us\big). \tag{10}$$

Model B As a little combinatorics shows, under model B, $P(c^{(51)} \stackrel{k}{\to} 51)$, equals

$$P\big(c^{(51)} \stackrel{k}{\to} 51\big) = \frac{2(N_{51} - [t^k_{51}N_{51}])}{N_{51}} \binom{N_{51}}{[t^k_{51}N_{51}]} 0.5^{N_{51}}$$
$$\approx \frac{2(N_{51} - t^k_{51}N_{51})}{N_{51}} \frac{1}{\sqrt{0.5\pi N_{51}}} \exp\left(-\frac{2}{N_{51}}\big(N_{51}t^k_{51} - 0.5N_{51}\big)^2\right), \tag{11}$$

where $[n]$ is the largest natural number $l \in \mathbb{N}$ so that $l < n$. The $P(c^{(51)} \stackrel{k}{\to} 51)$s go down very quickly, if one moves away from $t^5_{51} = 0.5$.

[10] For the determination of the thresholds see again http://www.lawanddemocracy.org/pdffiles/COamend36.pdf.

[11] The voting games $\mathcal{W}'_{51,k}$ cannot be assigned independent probabilities—they model the same voting. Composite voting games of this kind, where the assemblies are not pairwise disjoint, have recently been investigated by Edelman (2004). In his terms, $\mathcal{A}^{50}\mathcal{P}_C$ falls under the second variant of his model in Sect. 4. However, the structure of the assemblies is different between $\mathcal{A}^{50}\mathcal{P}_C$ and his example on pp. 225–227.

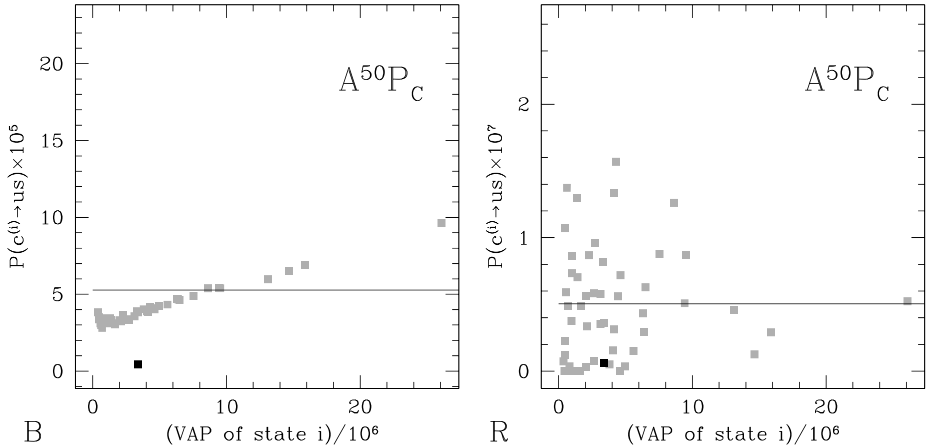

Fig. 2 The probability that a citizen $c^{(i)}$ is doubly pivotal under the voting scheme $A^{50}P_C$ as a function of VAP of her home state i. *Left panel*: model B. *Right panel*: model R. For more details see the caption of Fig. 1

The values of $P(51 \xrightarrow{k} us)$ are calculated using the voting power program. They are almost identical, and each of them is about a factor of 9 smaller than the voting power of Colorado's block vote under A^{51}. As $P(c^{(51)} \xrightarrow{5} 51) \gg P(c^{(51)} \xrightarrow{k} 51)$ for $k = 1, \ldots, 9$, $k \neq 5$, effectively only the threshold $t^5_{51} = 0.5$ matters, and we can approximate (10) by

$$P\big(c^{(51)} \to us\big) \approx P\big(c^{(51)} \xrightarrow{5} 51\big) \times P\big(51 \xrightarrow{5} us\big). \tag{12}$$

The voting powers for citizens from other states are calculated as follows. Under $A^{50}P_C$, Colorado's electors 1 trough 4 will almost certainly vote for X, whereas its electors 6 through 9 will almost certainly vote for Y. Elector 5 will vote for X with a probability of 0.5. So for each state $i \neq 51$, the power in the EC equals the power it would have within a variation of the EC, where Colorado has one instead of nine electors and the threshold for acceptance is reduced by 4 votes. The probabilities of being pivotal within those states, $P(c^{(i)} \to i)$ for $i \neq 51$ remain unaffected.

As the results in the left panel in Fig. 2 show, the powers of citizens from states different from Colorado only change minimally in comparison to A^{51} (left panel of Fig. 1). However, the voting power of Coloradans drops significantly. The relative share of power is now at 8.04%. This will be further discussed in Sect. 4.

Modell R For calculating the voting power of Coloradans under $A^{50}P_C$ for our model R, we can start from (9) again. However, unlike in (10), every summand $P(c^{(51)} \xrightarrow{k} us)$ now equals

$$P\big(c^{(51)} \xrightarrow{k} us\big) = \int d\mu_{us}\, p(\mu_{us}) P\big(c^{(i)} \xrightarrow{k} i|\mu_{us}\big) \times P(i \xrightarrow{k} us|\mu_{us}). \tag{13}$$

This is analogous to (5–6).

Generalizing (7), one can show that the first factor in (13) is about

$$P\big(c^{(i)} \xrightarrow{k} i|\mu_{us}\big) \approx \frac{p_{51}(\mu_i = t^k_{51}|\mu_{us})}{N_i} \tag{14}$$

(cf. Chamberlain and Rothschild 1981, p. 154). For the second factor in (13) again simulations are carried out. They operate exactly as they did for voting scheme A^{51}, except for the way the votes of the Coloradan electors are determined: Given a random realization of μ_{51}, we look for the largest $k \in \{1, \ldots, 9\}$ such that $\mu_{51} \geq t_{51}^k$. We let k electors vote for X and $(9-k)$ electors vote for Y. If $\mu_{51} < t_{51}^1$, all nine electors vote for Y.

Results are shown in the right panel of Fig. 2. Again there is not much change with respect to voting scheme A^{51} (right panel of Fig. 1). Only the voting power of Coloradans drops. Their relative share of power is now at 11.8%.

3.3 Proportional representation for all states (P^{51})

Suppose now that all states adopt proportional representation. What will be the effect for Coloradans and for citizens from other states?

Voting under this scheme can be modeled as

$$\mathcal{P}^{51} = \mathcal{V}''\big[\mathcal{W}'_{1,1}, \ldots, \mathcal{W}'_{1,n_1}, \mathcal{W}'_{2,1}, \ldots, \mathcal{W}'_{51,9}\big], \tag{15}$$

where $\mathcal{W}'_{i,k}$ is the voting game for the kth elector from state i—just as we had it for Colorado before. We assume that the thresholds t_i^k follow the same rule as applied in Amendment 36, i.e. $t_i^k = (k-0.5)/e_i$, where e_i equals the number of electors in state i. $\mathcal{V}''$ is a simple majority game with an assembly of size 538.

Model B Under the assumptions of Banzhaf voting power it is extremely likely that the fraction of votes for X is very close to 0.5 in each state. States i with an even number of electors, $e_i = 2l$ ($l \in \mathbb{N}$) will therefore almost certainly send l electors for X and Y each. As a result, their citizens will have no chance of being pivotal and thus have no voting power. States i with an odd number of electors $e_i = 2l+1$ ($l \in \mathbb{N}$) will almost certainly send l electors for X and l electors for Y. The remaining elector (for whom a threshold of $t_{51}^{l+1} = 0.5$ applies) will vote for X with a probability of 0.5. As for Colorado under $A^{50}P_C$, we need only consider this $(l+1)$th elector and the corresponding voting game $\mathcal{W}_{i,l+1}$ with a threshold of 0.5. The probability of being pivotal with respect to such an elector in one's home state is about

$$P\big(c^{(i)} \stackrel{l+1}{\to} i\big) \approx \sqrt{\frac{2}{\pi N_i}}. \tag{16}$$

Within the EC, most votes are almost certain, and these almost certain votes alone produce a draw. Thus only the 34 remaining electors from the 34 states with odd e_i have a fair chance of being pivotal within the EC. For estimating the power of such an elector, it is sufficient to consider simple majority voting with 34 voters. The voting power program yields a power of 0.136

Results are shown in the left panel of Fig. 3. Note that, whereas the present voting scheme A^{51} is more beneficial for larger states, P^{51} is more beneficial for smaller states, provided that they have an odd number of electors. Colorado is presently lucky to have an odd number of electors. Thus the relative share of power for a Coloradan is 186%.

Modell R For calculating the voting powers under the more realistic model, every state has to be dealt with in the way Colorado was handled under A^{51}. That is, the probability that a citizen is doubly pivotal is a sum of addends as in (13).

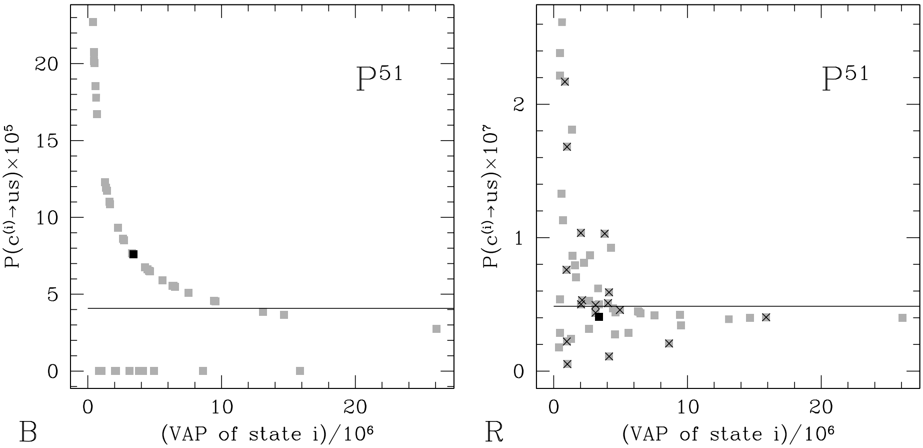

Fig. 3 The probability that a citizen $c^{(i)}$ is doubly pivotal under the voting scheme P^{51} as a function of VAP of her home state i. *Left panel*: model B. *Right panel*: model R. For more details see the caption of Fig. 1. In the *right panel*, states with an even number of electors are marked with a cross

Results are shown in the right panel of Fig. 3. As under model B, comparatively high voting powers are only obtained for citizens from small states. The explanation is that a citizen is typically more likely to be pivotal in a small state according to (14). Again, as under model B, citizens from states with an even number of electors (marked with a cross) have typically less voting power than citizens from states with an odd number of electors. However, there are some states i with an even e_i for which the citizens have a reasonable amount of voting power. The reason is that, under model R, the probability density for the fraction of votes for candidate X does not necessarily peak at 0.5 and it is not as sharply peaked as under model B. So there is a reasonable chance that the votes split exactly at threshold t_i^k for at least one or two values of k for some states i with an even e_i.[12]

Compared to P^{51}, the average voting power goes down. The relative share of power for a Coloradan is 83.6%.

4 Discussion

We now turn to a normative assessment of the three voting schemes under both models from the perspective of Colorado and from an impartial perspective. Our key quantities are summarized in Tables 1 and 2, for model B and R, respectively. The rankings obtained from the different perspectives under the different models are summarized in the matrix in Table 3. We will discuss each entry of the matrix in turn.

4.1 The Coloradan perspective

Let us first look at the left column of our matrix. The key quantities are presented in the first rows of Tables 1 and 2. The second rows present the relative shares of powers, but the normalization makes no difference to our analysis. The results are visualized in Fig. 4.

[12]In order to check whether our results in Sects. 3.1 through 3.3 are stable under small variations of the empirical input, we proceed as follows. Instead of fitting our model R to the past 10 elections, we fit it to a subset of 9 elections. We find that most of our results remain qualitatively stable.

Table 1 Results under model B. They were obtained from simulations. $\tilde{P}(c^{(i)} \to us)$ means the relative share of power

	A^{51}	$A^{50}P_C$	P^{51}
$P(c^{(51)} \to us)$	3.82×10^{-5}	4.25×10^{-6}	7.62×10^{-5}
$\tilde{P}(c^{(51)} \to us)$	7.19×10^{-1}	8.04×10^{-2}	1.86
$\sigma(P(c^{(i)} \to us))$	1.92×10^{-5}	2.01×10^{-5}	3.74×10^{-5}
$\sigma(\tilde{P}(c^{(i)} \to us))$	3.61×10^{-1}	3.82×10^{-1}	9.11×10^{-1}
$J^{DP}(\cdot)$	1.92×10^{-1}	2.04×10^{-1}	4.75×10^{-1}
Δ	1.10×10^{3}	1.12×10^{3}	1.89×10^{3}

Table 2 Results under the a posteriori model R

	A^{51}	$A^{50}P_C$	P^{51}
$P(c^{(51)} \to us)$	4.31×10^{-8}	5.92×10^{-9}	4.07×10^{-8}
$\tilde{P}(c^{(51)} \to us)$	8.60×10^{-1}	1.18×10^{-1}	8.36×10^{-1}
$\sigma(P(c^{(i)} \to us))$	3.59×10^{-8}	3.66×10^{-8}	3.00×10^{-8}
$\sigma(\tilde{P}(c^{(i)} \to us))$	7.17×10^{-1}	7.25×10^{-1}	6.18×10^{-1}
$J^{DP}(\cdot)$	3.86×10^{-1}	3.95×10^{-1}	2.42×10^{-1}
Δ	5.14×10^{4}	5.46×10^{4}	1.49×10^{4}

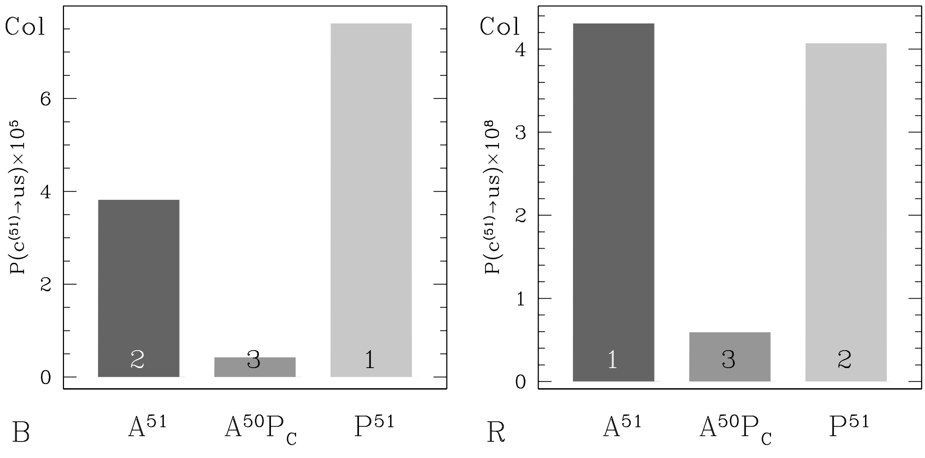

Fig. 4 Rankings of the voting schemes according to the voting power they confer to a Coloradan. The more power they confer, the better. *Left panel*: model B; *right panel*: model R

Unilateral switch from A^{51} *to* $A^{50}P_C$ Under the assumptions of Banzhaf voting power (*model B*), a unilateral switch to proportionality would be detrimental, since the power of a Coloradan would go down by a factor of 9, while the voting powers of other citizens would not be affected much.

The reason is as follows. Within the EC, the Banzhaf voting power of states is almost additive—i.e., if Colorado switches from a uniform block vote of 9 electors to 9 separate

Table 3 Ranking of the alternative voting schemes under both models (rows) from both perspectives (columns). $V_1 > V_2$ means that voting scheme V_1 scores considerably better than V_2; $V_1 \gtrsim V_2$ means that V_1 scores marginally better than V_2

Model	Colorado	σ^2, J^{DP}, Δ
B	$\mathrm{P}^{51} > \mathrm{A}^{51} > \mathrm{A}^{50}\mathrm{P}_{\mathrm{C}}$	$\mathrm{A}^{51} \gtrsim \mathrm{A}^{50}\mathrm{P}_{\mathrm{C}} > \mathrm{P}^{51}$
R	$\mathrm{A}^{51} \gtrsim \mathrm{P}^{51} > \mathrm{A}^{50}\mathrm{P}_{\mathrm{C}}$	$\mathrm{P}^{51} > \mathrm{A}^{51} \gtrsim \mathrm{A}^{50}\mathrm{P}_{\mathrm{C}}$

electoral votes, its power is almost evenly apportioned to these separate votes. Under the assumptions of equiprobable and independent votes within Colorado, the probability density that a fraction f of the Coloradan citizens vote for X is sharply peaked at $f = 0.5$. It is thus almost certain that at least four electors from Colorado will vote for X and that at least four of them will vote for Y. So effectively, the Coloradan vote commands only one elector, with one ninth of Colorado's power under A^{51}, and the power of a Coloradan decreases by a factor of 9.

Under the second, more realistic *model R*, a unilateral switch would also be detrimental for Coloradans. Their voting power would go down by a factor of 8. The reason is as follows. As under model B, the probability that Colorado's kth elector is pivotal in the EC, $P(c^{(51)} \stackrel{k}{\rightarrow} 51)$, is almost exactly 1/9 of the probability that the Colorado is pivotal in the EC under A^{51}. However, a Coloradan's voting power does not go down by a factor of nine, because the probability density that a fraction f of the Coloradan citizens votes for X is not as sharply peaked as under model B. There are two ties, viz. at t_{51}^4 at t_{51}^5, that have a non-negligible chance of occurring. The tie at t_{51}^5 did already matter under A^{51}. But the tie at t_{51}^4 becomes relevant only under $\mathrm{A}^{50}\mathrm{P}_{\mathrm{C}}$. It is because of this tie that the loss of voting power is smaller under model R than under model B, if we move from A^{51} to $\mathrm{A}^{50}\mathrm{P}_{\mathrm{C}}$.

This leaves us with a clear case against Amendment 36 from the perspective of Colorado. Regardless of the model that we assume, a significant loss of power would arise (see Fig. 4).

An objection At this point one might raise the following objection: for calculating the voting power of Coloradans under R, past data have been taken into account. However, the voting behavior of the US citizens is likely to change, if a new voting scheme is implemented, because voters make strategic choices that depend on the voting scheme adopted. Call this the *feedback effect*. If there is a feedback effect, then it is not realistic to adopt model R for $\mathrm{A}^{50}\mathrm{P}_{\mathrm{C}}$ and P^{51}.

To counter this objection, we calculate the power of a Coloradan under voting scheme $\mathrm{A}^{50}\mathrm{P}_{\mathrm{C}}$ under variations of model R, where we vary b_{51} by adding a $\Delta b_{51} \in [-0.1, 0.1]$. Accordingly, the expected fraction of votes for candidate X from Colorado varies. Our results are stable under this variation.[13]

Uniform switch from A^{51} *to* P^{51} If Colorado could convince every state to switch to proportionality, then the voting power of a Coloradan would increase relative to the present

[13] Of course, different feedback effects are possible. For instance, a_{51} might change, or the votes from other states might change as well. Now it may well be the case that one could do some fine tuning in the model so that the power of Colorado would actually go up. But then an argument would be needed that this fine tuning matches a reasonable expectation of a feedback effect.

system under *model B*. The reason is that the voting power of Colorado within the EC increases. Whereas, under the present scheme, it commands 9 out of 538 votes (i.e. 1.7 %), under P^{51} it effectively commands one out of 34 votes (i.e. 2.9 %).

Under *model R*, on the contrary, there would be a small loss of power, if Colorado could convince every state to follow suit. Due to the complexity of our model, there is no simple explanation of this fact. Note, also, that this result is not completely stable under variations of the input data. If one election is left out from the input data, sometimes a small gain in power can be observed. Moreover, this result might not be stable under a feedback effect. Unlike before, we cannot model the feedback effect by considering just one parameter—there might be different feedback effects in different states.

Putting these complications aside, for P^{51} the models yield different results. What model should we adopt to correctly assess voting schemes from the Coloradan perspective—model B or model R? Our argument proceeds in two steps.

First, Coloradans want to secure the influence they have in the real world and not under highly idealized conditions. They want to increase their influence given that the others vote as they do. Thus the influence they are concerned about is to be measured as the probability of being pivotal in the real world, and we should estimate voting power using our best estimate of what the future will be like.[14]

To illustrate this argument, let us suppose that Coloradans care about their voting power for instrumental reasons. E.g., they want to be more intensely exposed to campaigning, where more intensive campaigning in their state allows them to push their interests with the prospective President. One way to achieve this goal is to increase the voting power of the Coloradans, because both candidates are likely to allocate more resources to Colorado, if the Coloradans' power increases.[15] But if the candidates look at the voting power of the Coloradans for finding the best way to allocate their resources, they will base their judgment on the best evidence available. That is, they will work with the best estimate of what the voters' preferences will be.

Second, whether model B or model R is a better estimate for the future, depends on what exactly our task is. If our task is to design immutable institutions that will perform well for Colorado in the long run, we cannot rely on past preferences, because preferences are subject to change (see, e.g., Felsenthal and Machover 2000, p. 13). Thus, model B is more fitting and a switch to P^{51} is beneficial. On the other hand, if our task is to design institutions that will do well for Colorado in the predictable future and that can be updated when there is a substantial change in voting patterns, model R is more fitting, and a switch to P^{51} is not beneficial.[16]

[14]In model R, we assess whether Coloradans can increase their influence, given that all states, including Colorado, vote as they do. But one might object that Coloradans want to increase their influence, given that *other* states vote as they do and they themselves may vote differently after adopting new voting rules. This is a fair point and leads us back to our discussion of the feedback effect. Before we found that our results were stable under the feedback effect.

[15]More precisely, it is the relative share of power of Coloradans that matters, but we found identical rankings for power and relative share of power. Brams and Davis (1974, pp. 122–123) establish a connection between voting power and the resource allocation strategy they find to be optimal in a certain sense. For resource allocation strategies see also Colatoni et al. (1975), Crain et al. (1993), Grofman and Feld (2005).

[16]If there are significant feedback effects, then even the near future will be unpredictable.

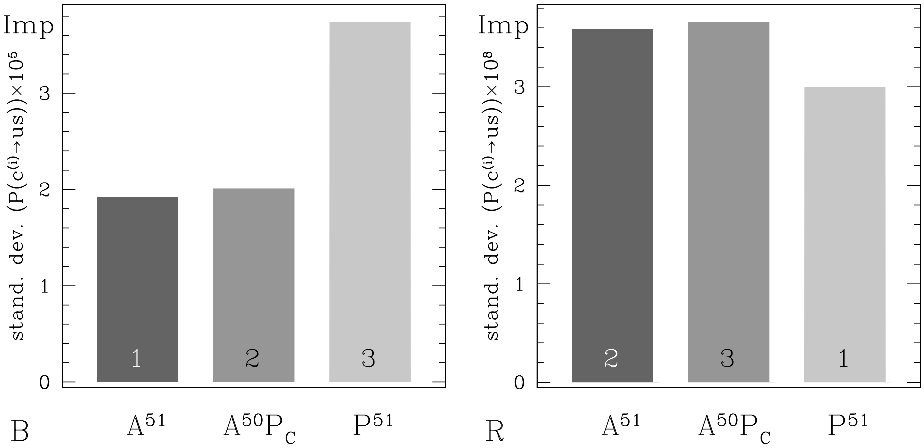

Fig. 5 The standard deviation of voting powers under A^{51}, $A^{50}P_C$ and P^{51} for model B (left panel) and model R (right panel)

Altogether, we conclude that, from the Coloradan perspective, there is a clear case against $A^{50}P_C$, whereas, whether P^{51} is beneficial depends on the task at hand.[17]

4.2 The impartial perspective

Let us now evaluate the alternative voting schemes from an impartial perspective. That is, let us turn to the right column in our matrix of assessments in Table 3. We had two desiderata, viz. to minimize the spread of the citizens' voting powers as measured by the standard deviation or J^{DP} and to minimize the MMD. The values of these quantities are summarized in Tables 1 and 2. As one can see, our desiderata for these quantities impose the same ranking of our voting schemes *within* each model. The rankings obtained for the standard deviation are visualized in Fig. 5.

Unilateral switch from A^{51} *to* $A^{50}P_C$ Under *model B*, a unilateral switch on the part of Colorado would impose slight cost on all desiderata. This is because Coloradans would lose voting power whereas the voting powers of citizens from other states would remain virtually unaffected. As a consequence, the Coloradans would become outliers and the spread of the distribution of powers would go up. This is reflected in both the standard deviation and J^{DP}. Because of the drastic drop in Coloradans' voting power, the voting system would also become less sensitive to the preferences of the voters, and the MMD would go up (see Theorem 3.3.17, p. 60, and Approximation 3.3.10, p. 56, in Felsenthal and Machover 1998).

Under *model R*, the present voting scheme is also slightly more responsive to the desiderata than after a unilateral change. The explanation hereof parallels the explanation under model B, as far as the spread in voting powers is concerned. There is no simple explanation why the MMD increases slightly.

[17] We have not considered a scenario, under which the number of electors changes for Colorado due to a change in population. If Colorado would gain another elector, its power would drop to virtually zero for P^{51} under model B.

Uniform switch from A^{51} *to* P^{51} If Colorado were to convince every state to follow suit, our desiderata would be met to a significantly less extent in *model B*. The reason is as follows. Under the present voting scheme, the loss of intrastate voting power as a function of population is partly offset by the fact that larger states command more electors. But in the new system, all states with an odd-number of electors effectively command the same number of electors (viz. one). Hence the spread between the voting powers of their citizens will be large. Furthermore, in the new system, states with an even number of electors effectively command zero electors. This makes the spread even larger. Since voters from states with an even number of electors have no power under P^{51}, the sensitivity is small. Under the assumptions of Banzhaf voting power it follows that the MMD is comparatively large.

However, under *model R* the resulting system would score slightly better on the desiderata than the present system, if Colorado could convince everyone to follow suit. Once again, there is no simple explanation of this fact. This ranking comes about due to a complex interplay of various factors.

So for P^{51} the models yield different results: A switch from A^{51} to P^{51} would be beneficial from an impartial perspective under R, but detrimental under B (see Tables 1 and 2 and Fig. 5). So what model should we base our impartial ranking on? There are two issues that are relevant to this question.

The first issue is about modeling. As before (p. 243) if our task is to design immutable institutions that will perform well for the US in the long run, then model B is more fitting. On the other hand, if our task is to design institutions that will do well for the US in the predictable future, model R is more fitting.

The second issue is as follows. Felsenthal and Machover (2000, p. 13) argue that an impartial assessment of a voting scheme requires one to put oneself behind the Rawlsian veil of ignorance (see Rawls 1971) and that the veil of ignorance excludes knowledge of preferences. Clearly, impartiality requires that as a citizen of a state I should not choose institutions by taking into account the interests of my state. But why should impartiality require that I should choose institutions abstracting from the profile of the different states' interests?

Nonetheless, Felsenthal and Machover may be correct in their resistance to import empirical information about interest profiles. There does seem to be something untoward in tuning a voting scheme to the preferences of the voters. A state should not receive more voting power, because the voting profile of the citizens is such that the chance of being doubly pivotal is substantially lower than in other states (Gelman et al. 2004, p. 671).

We leave it as an open question what kind of information one should take into account when assessing whether one voting scheme is more beneficial than another from an impartial point of view. This is why we chose to analyze both models.[18]

4.3 Summary

Although we could not provide a final answer to all questions that we have raised, our analysis leaves us with a number of definite results. Let us finally put together both perspectives.

1. From both the Coloradan and from an impartial perspective, there is a strong case against a unilateral switch to Amendment 36. Prudence on the part of Colorado and morality agree. This holds independently of the model assumed.

[18]The situation is even more complex. Recently, a model has been proposed that strikes an interesting compromise between model R and model B. Gelman et al. (2004) propose to neglect particular information about voter preferences but to include the pattern of preferences. That is, equiprobability is granted but independence is not.

2. A collective switch to proportional representation (P^{51}) would pay off from the Coloradan perspective under the assumption of Banzhaf voting power, but not under the assumption of an a posteriori model. Such a switch would pay off from an impartial perspective under the assumption of an a posteriori model, but it would be detrimental under the assumption of Banzhaf voting power.

Finally, even if some people might want to contest our normative claims, our paper illustrates the difficulties one faces in assessing a voting scheme, and our results under model R could be used for predictive purposes—at least for the predictable future.

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