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Minimizing the threat of a positive majority deficit in two-tier voting systems with equipopulous units

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Abstract The mean majority deficit in a two-tier voting system is a function of the partition of the population. We derive a new square-root rule: For odd-numbered population sizes and equipopulous units the mean majority deficit is maximal when the member size of the units in the partition is close to the square root of the population size. Furthermore, within the partitions into roughly equipopulous units, partitions with small even numbers of units or small even-sized units yield high mean majority deficits. We discuss the implications for the winner-takes-all system in the US Electoral College.

Keywords Two-tier voting system · Mean majority deficit · Voting power · Electoral College · Sensitivity · Majoritarianism

1 Introduction

A vote is taken in a population—in a company, in a university, in a nation or in a federation of states. A binary issue is on the table, say whether to adopt a certain strategy or not, whether to elect one of two candidates for president, and so on. One may decide the issue by means of a simple majority vote. As is well known (May 1952), simple majority voting is uniquely characterized by a number of requirements that seem reasonable for many voting systems. However, there may be historical or other reasons to organize the vote as a *two-tier* vote. In a two-tier voting system, the population is split into smaller units. A vote is taken in each unit, a representative of the unit will convey the outcome of the majority vote in her unit to a board of representatives, and a vote in that board will decide the issue. The latter procedure may have certain advantages. To name one such advantage, people

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may identify with their unit and may feel less alienated from the political process if their vote is recorded at the level of their unit and if this vote carries weight at the level of the board. But there is also a clear disadvantage. A two-tier voting system may yield a different outcome from the outcome of the *popular vote*—a proposal may be accepted, say, although a majority of the people voted against it. We say that there is a *positive majority deficit* iff (if and only if) more than the majority of the people cast a vote that is different from the outcome.

Such a positive majority deficit raises questions of legitimacy. How can a democratic decision neglect the wishes of the majority of the people? This was precisely the problem of the 2000 US presidential elections when Bush won even though Gore received 543,895 more votes than Bush in the popular vote.[1] This number of excess votes is the majority deficit. It is thus plausible to raise a *majoritarian concern* for voting rules. When discussing the Council of the European Union, Felsenthal and Machover (2000: 18) put the concern thus: "the rule used by the council should arguably come as close as possible to producing outcomes that conform to the wishes of the majority of the entire electorate."

The majoritarian concern may be explicated in different ways. One idea is to say that a positive majority deficit should be avoided as often as possible, whatever its size might be. In this case we should minimize the *probability of a positive majority deficit*. In this spirit, Hinich et al. (1975) consider the probability of a positive majority deficit for a toy model of the United States under which the US states are equipopulous. However, we find it more plausible to assume that, if there is a positive majority deficit, then it is better for it to be minimal. This leads to another explication of the majoritarian concern (Felsenthal and Machover 2000: 23–25). The desideratum then is to minimize the *mean majority deficit* (MMD), i.e., the expectation value of the majority deficit (see below for a more precise definition). Thus, in this paper, we will study the MMD.

Two-tier voting encompasses a broad range of voting procedures and in principle there are many characteristics we may want to vary to minimize the MMD. In this work, we assume that we have a free hand in splitting up the population into units and stipulate that the units should be equal-sized or almost equal-sized. We need to decide into *how many* units we will split the population. We will monitor the MMD and ask: How does our choice of the number of units affect the MMD?

For simplicity, we will keep other characteristics of the two-tier voting system fixed. In most of our calculations, we will assume that there is simple-majority voting at both tiers of the voting system—in the units and in the board of representatives. This choice is in fact very good for minimizing the MMD.

For calculating expectation values, we will adopt the Bernoulli model as a probability model, i.e., each voter is equally likely to vote one way or the other and there is probabilistic independence between the votes. Our assessment of different partitions is thus *a priori* because no empirical information enters. Beisbart and Bovens (2008) consider *a posteriori* estimates of the mean majority deficit for the US Electoral College.

Much of the existing research literature takes a perspective that is different from ours. Felsenthal and Machover (1998, Sect. 3.4) and Felsenthal and Machover (1999) assume that the partition of the population is given. Units are in fact often pre-existing, e.g., in a federation of independent states such as the European Union. Felsenthal and Machover search for a voting procedure for which the MMD is minimal under the Bernoulli model. They assume that there is simple majority voting in the units and find for arbitrary partitions that the

[1] http://www.infoplease.com/ipa/A0876793.html (checked May 2011).

MMD is minimal iff the representatives vote following a weighted voting rule under which the weight of a unit is proportional to the square root of its population. This result is sometimes called the second square-root rule. Felsenthal and Machover (2000) study alternative voting rules for the EU Council of Ministers in light of the MMD (and of a few other concerns). Our work builds upon theoretical results summarized in Felsenthal and Machover (1998), but our setting differs from theirs in two ways. First, we do not take the partition into units as fixed. Second, our focus is on partitions with equipopulous units or almost equipopulous units.

We will proceed as follows. We first explain the theoretical framework upon which our results are based (Sect. 2). We then investigate how the MMD behaves for partitions with exactly equipopulous units in Sect. 3. Subsequently we move to partitions with quasi-equipopulous units, i.e., partitions in which the population number n is not divisible by the number of units, m, but the partitions deviate only minimally from an equipopulous partition (Sect. 4). In Sect. 5 we consider partitions that allow for larger deviations from equipopulous units. We conclude with a discussion of the political relevance of our findings for the design of two-tiered voting systems, in particular for the US Electoral College (Sect. 6).

2 Theoretical framework

In this section, we will briefly explain the theoretical framework in which we will obtain our results. This section is kept as general as possible. More specific assumptions are adopted, as the paper proceeds.

Consider a population of n voters. The population is split into m units, where $1 \leq m \leq n$. The units are numbered from $j = 1$ to $j = m$; for each j in this range, the j-th unit has n_j voters. We will later assume that the sizes of the populations are equal or almost equal. Clearly, $\sum_{j=1}^{m} n_j = n$.

A binary issue is on the table, call the options Yes and No. Each unit has one representative, and her vote is the outcome of a simple majority vote within her unit. The final outcome of the vote is x iff more than one half of the representatives vote x for $x =$ Yes, No in the board of representatives. Such a voting system can be modeled in terms of a composite voting game (Felsenthal and Machover 1998, Def. 2.3.12 on p. 27). Note that there need not be actual representatives that convey the votes of the units.

The majority deficit MD for a particular vote equals zero when the outcomes of the popular vote and the two-tier voting system coincide and it takes the positive value l when the outcomes disagree and there is a margin of l voters. Here, the outcome of the popular vote is the outcome that would arise under simple majority voting. The MMD is the expectation value $E[\text{MD}]$ of the majority deficit. That is, the MMD equals the sum of the products of the probability that the majority deficit takes on the value MD times MD (Felsenthal and Machover 1998: 60). In this paper, we will assume the Bernoulli model as a probability model in order to calculate expectation values. That is, the votes of the citizens are probabilistically independent and each citizen has a probability of 0.5 of voting Yes.

Suppose now that $m = 1$ or that $m = n$. These are clearly limiting cases of two-tier voting procedures, which just coincide with a popular vote or with simple majority voting. Hence, the outcome of the vote necessarily reflects the majority of the votes; there is no threat of a positive majority deficit, and the MMD equals 0. But suppose there are reasons to have a non-trivial two-tier voting system, for which m is strictly between 1 and n. How does the MMD behave, as m is being varied? What is the m-value that yields a maximal MMD and is thus worst? That is the central question of this paper.

Our results rest on a theorem in the literature about voting power. We follow Felsenthal and Machover (1998: 60), who make reference to Dubey and Shapley (1979). Assuming the Bernoulli model, the theorem relates $E[\mathrm{MD}]$ to *sensitivity* S, which is defined as the sum of the non-normalized Banzhaf voting powers β_i' for all voters $i = 1, \dots, n$ (Felsenthal and Machover 1998: 39). According to the theorem, $E[\mathrm{MD}]$ is a linear transform of S:

$$E[\mathrm{MD}] = \frac{S_n - S}{2}, \tag{1}$$

where S_n is the sensitivity for simple majority voting with n voters and is a constant for a given n. The theorem implies that maximizing sensitivity is equivalent to minimizing $E[\mathrm{MD}]$. Since the optimal value for $E[\mathrm{MD}]$ is zero, maximal sensitivity obtains for simple majority voting when S equals S_n.

Let us now calculate the sensitivity S for our two-tier voting system. The voting power of a single citizen is the probability that her vote is doubly pivotal. A vote of a single citizen is doubly pivotal iff the outcome of the two-tier vote would have been different had the vote been different. Double pivotality requires that the vote is pivotal in its unit and that the representative of this very unit is pivotal in the board of representatives. Under the Bernoulli model, the probability of double pivotality factors in the probability that the vote is pivotal in its unit and that the respective representative is pivotal in the board. The probability that a citizen's vote is pivotal in a simple majority vote with k voters, P_k, is

$$P_k = \binom{k-1}{[k/2]} \Big/ 2^{k-1}, \tag{2}$$

where $[k]$ is the largest integer l with $l \leq k$ and $\binom{\cdot}{\cdot}$ is the binomial coefficient. The probability that a voter in unit j is pivotal in her unit (*unit*) thus equals

$$P_j(\mathit{citizen} \rightarrow \mathit{unit}) = \binom{(n_j - 1)}{[n_j/2]} \Big/ 2^{n_j - 1}. \tag{3}$$

Call the probability that a representative of unit j is pivotal in the board of representatives (*br*) $P_j(\mathit{unit} \rightarrow \mathit{br})$. In the special case in which the representatives vote by the way of simple majority voting, this probability equals

$$P_j(\mathit{unit} \rightarrow \mathit{br}) = \binom{m-1}{[m/2]} \Big/ 2^{m-1}. \tag{4}$$

Thus, the voting power of a voter from the j-th unit is

$$P_j(\mathit{unit} \rightarrow \mathit{br}) \times \binom{(n_j - 1)}{[n_j/2]} \Big/ 2^{n_j - 1} \tag{5}$$

and the sensitivity equals

$$S = \sum_{j=1}^{m} n_j \times P_j(\mathit{unit} \rightarrow \mathit{br}) \times \binom{(n_j - 1)}{[n_j/2]} \Big/ 2^{n_j - 1}. \tag{6}$$

The sensitivity for the popular vote is n times P_n as defined in (2). The result is

$$S_n = n \times \binom{n-1}{[n/2]} \Big/ 2^{n-1}. \tag{7}$$

From Felsenthal and Machover (1998: 56), we know that S_n approaches $\sqrt{2n/\pi}$ for large n. We will sometimes use this approximation. Not much hinges on that because our primary concern—the m-dependence of $E[\text{MD}]$—is not affected by that approximation. Combining (1) and (3)–(6), we can evaluate the mean majority deficit for every pair of n- and m-values.

3 The MMD for exactly equipopulous units

In this section, we assume that the population can be split into exactly m units with exactly n/m people, each, where both m and n/m are integer-valued. Of course, for a given number n, there will not be many ways to split the population in this way. In this case, each citizen has the same voting power of

$$\beta' = \binom{(n/m-1)}{[n/(2m)]} \times \binom{m-1}{[m/2]} \Big/ \left(2^{n/m-1} \times 2^{m-1}\right) \tag{8}$$

and the sensitivity is n times this expression. The numerical value of the sensitivity measure depends on what the [ ]-brackets yield, and this depends on whether n and m/n are even or odd. The reason is as follows: Assume that k is even. In a simple majority vote with k voters, at least $(k/2+1)$ voters have to vote Yes for acceptance. When an additional voter is added to yield an electorate of $(k+1)$ voters, it is still at least $(k/2+1)$ voters that have to vote Yes for acceptance. Thus, if normalized by population size k, the threshold of acceptance is higher when k is even. This has an important consequence for the probability that a vote is pivotal under simple majority voting. Generally, this probability decreases, as the population size k becomes larger. However, when we move from an even k to $(k+1)$, the mean majority deficit stays constant. This will become important later in the paper.

We thus have to distinguish between several cases. It is first useful to discuss the cases in which n is odd or even.

3.1 First case: odd n

If the population size is odd, the numbers of units and the numbers of voters within each unit both have to be odd as well. As a consequence, the sensitivity is

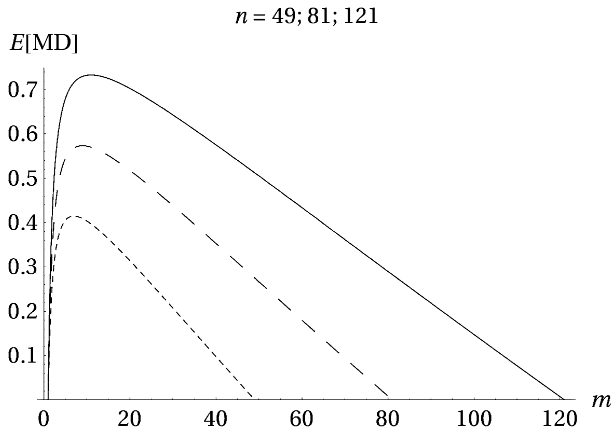
$$S = n \times \binom{n/m-1}{(n/m-1)/2} \Big/ 2^{n/m-1} \times \binom{m-1}{(m-1)/2} \Big/ 2^{m-1}. \tag{9}$$

Strictly speaking, this result holds only for m and n/m being integers. But if we express the binomials in terms of factorials and use the Gamma function as an extension of the factorial ($n! = \Gamma(n+1)$), we may think of the voting power and of $E[\text{MD}]$ as functions that are defined on the whole interval $[1, n]$. Call these functions the *real-extended* voting power and the *real-extended* $E[\text{MD}]$. We will more generally say that a function is real-extended if its domain is extended from a range of natural numbers to a range of real numbers by replacing factorials using the Gamma function.

In Fig. 1, we show the real-extended $E[\text{MD}]$ for three different values of n as a function of m.

The curves for the different n-values have similar shapes: They start at 0, increase as m increases, reach a maximum and decrease to zero again. For each curve, the maximum is located at $m = \sqrt{n}$. The same holds for other values of n, even for n-values that cannot be thought of as the square of an integer.

Fig. 1 The real-extended mean majority deficit $E[\mathrm{MD}]$ for two-tier voting systems with $n = 49$ (*short-dashed*); 81 (*long-dashed*); 121 (*solid*) voters as a function of the number of the units m



How can we understand this result? We have not been able to prove analytically that, for any n, the curve representing the MMD follows the shape that can be seen in Fig. 1. But we can provide some analytic results that help us understand and generalize what we observe in Fig. 1.

Due to (1), the real-extended $E[\mathrm{MD}]$ is a linear transform of sensitivity, which, in our case, is n times the probability of double pivotality.[2] Thus, the real-extended $E[\mathrm{MD}]$ is a linear transform of the voting power of a citizen. This probability can be written as a product of the following form:

$$g(m) \times g(n/m), \tag{F}$$

where g is some differentiable function (cf. (8)). Consequently, the value of the sensitivity and thus of the MMD does not change when we set m at $m' = n/m$. The reason is that it does not matter whether we have m units of n/m people each, or n/m units of m people each. In both cases, the probability that a citizen is doubly pivotal is the same product of two probabilities that a voter is pivotal in simple majority voting, once in an electorate of m people, and once in an electorate of n/m people. (This is also true in case n is even.)

The derivative of (F) equals

$$g'(m) \times g(n/m) - \frac{n}{m^2} g(m) \times g'(n/m). \tag{10}$$

It is easy to see that this derivative is zero at $m = \sqrt{n}$ and that, if the derivative has a certain sign in the interval $[\sqrt{n}, \sqrt{n} + \varepsilon]$ for some positive ε, it has the opposite sign in the interval $[\sqrt{n} - \varepsilon, \sqrt{n}]$. This shows that the probability of double pivotality and thus the real-extended $E[\mathrm{MD}]$ has an extremal value at $m = \sqrt{n}$ (unless it is a straight line in some range around $m = \sqrt{n}$). By plotting the second derivative of the real-extended $E[\mathrm{MD}]$ at $m = \sqrt{n}$ for a broad range of values of n, we observe that the extremal value is a maximum.[3]

But that does not yet settle the question as to how $E[\mathrm{MD}]$ behaves as a function of m because there may be other minima or maxima. In particular, we cannot yet infer that the

[2] Strictly speaking, we mean the real-extended probability of pivotality, the real-extended sensitivity, but for simplicity we will sometimes drop the "real-extended".

[3] Using Rolle's Theorem, one can show more generally that any function h for which $h(m) = h(m/n)$ has a zero derivative at $m = \sqrt{n}$.

maximum that we have found is a global one in the interval $[1, m]$. In Appendix 1 we use approximations to strengthen our results.

Our main result from this subsection—viz. that the MMD becomes maximal for the square root of n—may be named a 'square-root rule'. In the literature, a number of square-root rules have been found so far. We clarify the significance of our result by comparing it to three other square-root rules in voting theory.

The so-called first square-root rule concerns Banzhaf voting power in a two-tier voting system with units that are large in population, but not necessarily equipopulous. According to the first square-root rule, the voting powers of the people are equalized if the weights of the units in the board of representatives are set proportional to the square roots of the population sizes (see Felsenthal and Machover 1998: 66–68 for a statement of the rule and further references, e.g., Penrose 1946 and Banzhaf 1965; see Felsenthal and Machover 2000 and Życzkowski and Słomczyński 2004 for recent applications to the Council of Ministers in the European Union). Clearly, our result differs from the first square-root rule since we are not here concerned with equalizing voting power.

The second square-root rule has already been mentioned; it states that the MMD is minimal in a two-tier voting system with sufficiently large units iff the weights for units in the board of representatives are proportional to the square roots of the population sizes (see Felsenthal and Machover 1998: 74–75 for a discussion). Although the second square-root rule and our result both concern the MMD, there are significant differences. Whereas the second square-root rule aims to determine the weights for a voting system with fixed units of various sizes in order to minimize the MMD, our result concerns a partition into equipopulous units that maximizes the MMD.

Recently, a third square-root rule has been proven by Edelman (2004, see also Edelman 2005 for further discussion). Edelman considers two-tier voting systems under which a voter has several votes that may be cast independently. Since these voting systems are markedly different from the voting systems that we investigate, there is no connection between our results and Edelman's square-root rule.

Note, however, that care is required in interpreting our results. In our figures and arguments, we take $E[\text{MD}]$ to be a continuous function of m. But, ultimately, m can only take integer values, and the continuous curves are only extrapolations. Moreover, m has to be chosen in such a way that n/m is an integer. Suppose now that somebody wants to split an even-sized population into exactly equipopulous units, while *maximizing* the MMD (which is, of course, not recommended). Our results suggest that there should be $m = \sqrt{n}$ units, but $\sqrt{n}$ will in general not be an integer, and so splitting the population into $\sqrt{n}$ units is not an option. Instead, to maximize the MMD, one would have to choose the m-value m^* that is a divisor of n and that is closest to $\sqrt{n}$. And of course n/m^* yields the same maximal MMD.

3.2 Second case: even n

Let us now consider the case of even population sizes. If we partition the population into exactly m units, m may be even or odd. Thus, the following three sub-cases arise:

2a. n even, m even and n/m odd. This covers the special case of $m = n$. From (8) we obtain the following expression for the voting power of an arbitrary citizen in this subcase

$$\beta' = \binom{m-1}{m/2} \times \binom{n/m-1}{(n/m-1)/2} \times \frac{1}{2^{m-1}2^{n/m-1}}$$

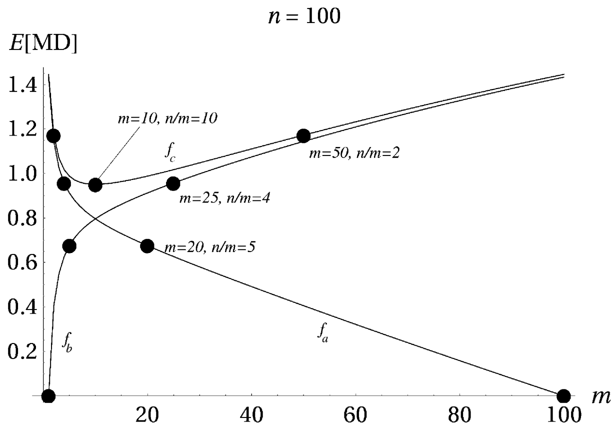


Fig. 2 The real-extended E[MD] for even n and the sub-cases 2a (m even and n/m odd), 2b (m odd and n/m even) and 2c (m even and n/m even). The *dots* denote the divisors of $n = 100$ and the corresponding values of E[MD]

2b. n even, m odd and n/m even. This covers the special case of $m = 1$. The voting power of a citizen is now

$$\beta' = \binom{m-1}{(m-1)/2} \times \binom{n/m-1}{n/2m} \times \frac{1}{2^{m-1}2^{n/m-1}}$$

2c. n even, m even and n/m even. The voting power of a citizen is now

$$\beta' = \binom{m-1}{m/2} \times \binom{n/m-1}{n/2m} \times \frac{1}{2^{m-1}2^{n/m-1}}.$$

Note that sub-cases 2a and 2b will occur only for every even n, while sub-case 2c will only occur for n-values that are multiples of 4. Note also that the expressions for voting power are slightly different.

From the probability of double pivotality, E[MD] can easily be calculated for each sub-case using (1) and (3)–(6). Accordingly, we obtain a slightly different expression for E[MD] for each sub-case. Each of these formulae can be real-extended. Call the real-extensions of E[MD] for the three sub-cases $f_a(m)$, $f_b(m)$ and $f_c(m)$, respectively. Strictly speaking, these functions also depend on the value of n, but for simplicity of notation, we drop the n. We plot the real-extensions for $n = 100$ in Fig. 2.

Because of the alternative sub-cases, the maximal MMD can not be derived by analyzing one real-extension. But looking at the blacks dots, which refer to the partitions into equipopulous units, we learn that the MMD is maximal iff there are two units or the units have two people, each.

Let us now provide some general results. First, in case (c), the probability of double pivotality is a product of the following form:

$$g(m) \times g\left(\frac{n}{m}\right). \tag{F}$$

Following our reasoning above, we can infer that $f_c(m)$ has a zero derivative at $m = \sqrt{n}$. This time, it is apparently a minimum, at least for the examples that we have considered.

The definitions of the first two curves imply that they stand in the following relation:

$$f_a(m) = f_b\left(\frac{n}{m}\right). \tag{11}$$

This means that it does not matter whether we partition the population into m or into n/m units—the E[MD] is the same for both cases. An immediate consequence is that $f_a(m)$ and $f_b(m)$ intersect with each other at $m = \sqrt{n}$. So that point is once again important, but this time not because it is a maximum.

The next question is whether there is a general conclusion regarding the value of m for which the E[MD] is worst (i.e., maximal). We observe that, for each n and each m, $f_c(m)$ always lies above the curves for $f_a(m)$ and $f_b(m)$. It can in fact easily be shown analytically that, for $4 \le m \le n/4$, $f_c(m)$ is higher than the curves for $f_a(m)$ and $f_b(m)$.[4] We will now assume that this is so in the whole relevant range of m-values. We can thus infer the following:

a. If n is a multiple of 4, the third curve $f_c(m)$ is relevant. Because of our observation, the maximal E[MD] must lie on this curve or below it—the curve is an upper bound for the maximal E[MD]. This curve has a minimum at $m = \sqrt{n}$. Therefore, the further we move away from $m = \sqrt{n}$, the larger is $f_c(m)$. Now the furthest we can get from that point with a permitted m-value (m being a divisor of n), is $m = 2$ or $m = n/2$. Since n is a multiple of 4, we know that E[MD] coincides with the curve at this point. Thus, $m = 2$ or $m = n/2$ provide the maximal E[MD].
b. If n is not a multiple of 4, $f_c(m)$ is not relevant, and we have to work with the other two curves. We have found that, for a number of suitable values of n, the curves for $f_a(m)$ and $f_b(m)$ always have shapes similar to those in Fig. 2. We conclude that $m = 2$ or $m = n/2$ provide the maximal E[MD] in case n is not a multiple of 4.

Combining these results, we can formulate a rule that holds for all even n: If n is even, then the maximal E[MD] is located at $m = 2$ and $m = n/2$.

How can we explain the validity of this rule? For a vote with two people, simple majority voting effectively requires a unanimous Yes if the outcome is to be Yes. Thus, if a two-tier voting system has units of two people each, then there are quite a lot of possible voting profiles under which the outcome is No, although the majority of the people voted Yes. This effect will make a considerable contribution to the MMD.

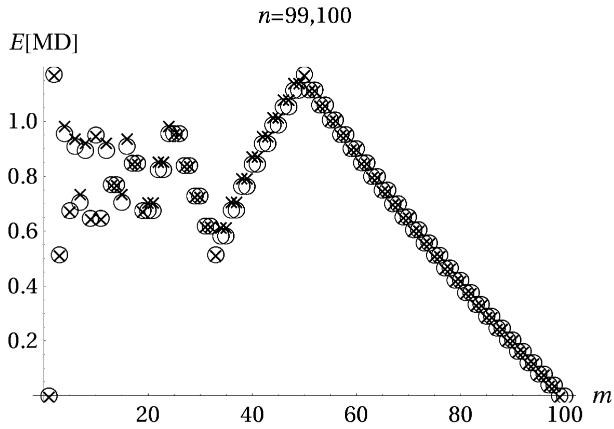
As is well-known, simple majority voting with an even number of voters is not a proper voting game (see Felsenthal and Machover 1998: 11 for the definition of a proper voting game). Accordingly, neutrality is violated (cf. May 1952), and, under the Bernoulli model, the outcomes Yes and No are not equally likely. The same is true for our two-tier voting systems with even n. A possible criticism is thus that the two-tier voting systems that we investigate for even n do not make much sense.

The problem can be avoided if we replace simple majority voting with the following voting system: The result of the vote is the same as under simple majority voting unless there is a tie. Whenever there is a tie—whenever there are as many Yes-votes as No-votes—a fair coin is flipped, and the outcome is Yes (No) with a probability of 0.5 (0.5). From a theoretical point of view, such a voting system is described as a lottery of voting games (Laruelle and Valenciano 2004: 418).

Let us therefore consider our two-tier voting system and replace each simple majority vote with an even number of voters by a lottery of voting games as described above. That is, both at the level of the units and at the level of the board of representatives, simple majority

[4]To show this, one has to consider the pertinent expressions for the probability of double pivotality, to take their logarithms and to exploit the fact that the logarithm of the gamma function is a convex function for arguments no smaller than 2.

Fig. 3 The E[MD] as a function of m for $n = 99$ (*crosses*) and $n = 100$ (*circles*) for quasi-equipopulous units



votes with even numbers of voters are replaced in that way. The result can again be described as a lottery of voting games.

In Appendix 2 we analyze such a voting system. We show that the value of the MMD does not at all differ from the results that we have obtained for even numbers of voters. Our results obtained so far for even numbers of votes are thus not artefacts that can arise only for improper voting games.

4 Partitions into units that do not have exactly the same size

So far, we have only considered partitions with units that are exactly equipopulous. But if the population size n is a prime number, there will be no way to obtain such a partition. And even in case n is not prime, there will not be many ways to partition the population into exactly equipopulous units. We therefore consider partitions with units that are not exactly equipopulous, but that display only minimal deviations from equipopulous units.

Start with a population of n people. For any integer $m = 1, \dots, n$, we can construct a partition such that all units have between $[n/m]$ to $([n/m]+1)$ people, where $[j]$ is again the largest integer smaller than or equal to j. For $n = 10$ and $m = 4$, for instance, we partition the population into two units with two voters each, and two units with three voters each. Given values of m and n, the distribution of the population sizes of the units is uniquely fixed. Let us call these partitions quasi-equipopulous.[5]

In case the units are not equipopulous, the question arises as to how we assign weights to representatives of each unit. We will consider two methods—either we continue to assign equal weights to the representatives, or we assign weights to the representatives such that the MMD is minimized.

Let us first consider the first method. Although the units are no longer equipopulous, their representatives are assigned equal weights. Results for $n = 99$ and $n = 100$ can be seen in Fig. 3. We have obtained similar results for other values of n.

We observe that the curves are similar for neighboring values of n. This should not come as a surprise, for how can one additional voter make a tremendous difference for minimizing

[5] In our usage of the term, we will allow that a quasi-equipopulous partition is exactly equipopulous in special cases.

the MMD if n is very large? For small numbers of units, m, the MMD jumps back and forth. The pattern here is roughly as follows: For even values of m, the MMD is comparatively large, whereas it is comparatively small for odd values of m. For larger m-values, the MMD decreases or increases between more pronounced extremal values. Local maxima of the MMD arise for partitions in which there are (almost) equipopulous units with even values of n/m. For instance, there is a maximum at $m = 50$, where $n/m = 2$ for (almost) every unit. Minima arise for partitions in which there are (almost) equipopulous units with odd numbers of n/m. For instance, there is a minimum at $m = 33$, where $n/m = 3$ for (almost) every unit. For $m > n/2$, the MMD decreases and approaches zero, as m increases—the more units there are with one voter each, the smaller is the MMD, since we approach the popular vote. The general moral is clear. In the design of boards of representatives, small even-sized boards and small even-sized units yield high MMDs; small odd-sized boards and small odd-sized units yield low MMDs. These results are not affected if simple majority votes with even numbers of votes are replaced by lotteries of voting games as indicated above.

But why does a small even-sized number of units yield a high MMD? As we have mentioned above, in a population with k voters, the probability of pivotality under simple majority voting stays the same when we start with an even k and move to $(k + 1)$. Suppose now that our population has $n = 99$ people and is split into $m = 4$ units. That is, we have 4 units with about 25 voters, each. Assume now, we re-partition the population into 5 units. We know that the voting power of a representative in the board will not change, as we move from 4 to 5 representatives. However, the units are significantly smaller under $m = 5$ than they were under $m = 4$. As a result, the probability that a citizen is pivotal in her unit will increase. Consequently, the voting power of each citizen, which is a product of the voting power of her representative and the probability that she is pivotal in her unit, will increase. So will the sensitivity, and, hence, the MMD will decrease. This decrease is particularly significant if m is small because, in this case, the probability that a citizen is pivotal in her unit will grow more significantly than if m is larger. This explains why a small number of even-sized units are fairly bad and why the MMD jumps back and forth for small values of m.

How do the results in Fig. 3 relate to our earlier results for exactly equipopulous units? In the left panel of Fig. 4, we show the results for $n = 99$ and the real-extended curve for equipopulous units. We use thick dots to mark the MMD at m-values for which we have exactly equipopulous units. Of course, each dot lies on a curve of a real-extended MMD for equipopulous units. Everywhere else, the real-extension yields a lower bound on the MMD for quasi-equipopulous units.

In the right panel of Fig. 4, we show the results for $n = 100$ and the real-extensions for the three sub-cases that we distinguished in Sect. 2. The real-extension of the curve for an even number of equipopulous even-numbered units is now an upper bound on the values of the MMD for the quasi-populous units.

Let us now consider the second method of assigning weights to the units. The idea is to find a voting rule in the board of representatives such that the MMD is minimized. Equivalently, we want to have a rule that maximizes sensitivity (cf. (1)). To find such a rule, we take the partition as given and the rule of the board as variable so that we may change it to maximize sensitivity. The sensitivity can be written as a weighted sum of the voting powers of the representatives,

$$S = \sum_{j=1}^{m} n_j \times P(citizen \to unit\, j) \times P(unit\, j \to br) = \sum_{j=1}^{m} c_j(n_j) \times P(unit\, j \to br). \quad (12)$$

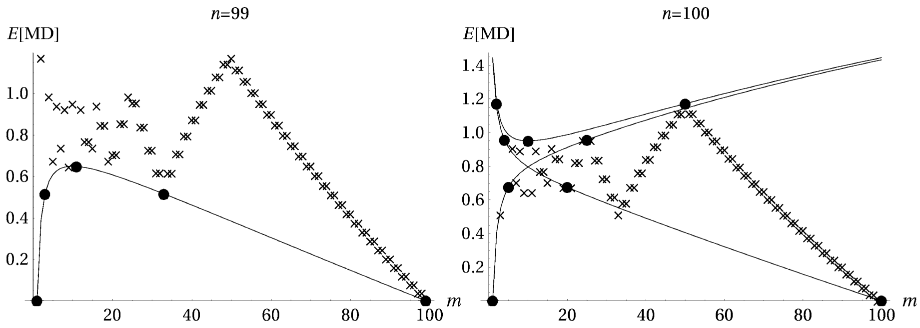
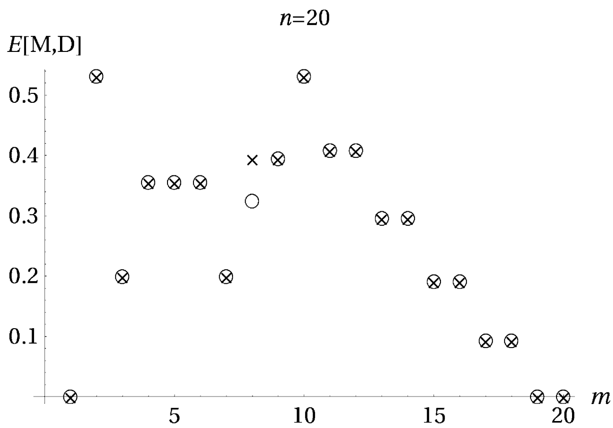


Fig. 4 The $E[\text{MD}]$ as a function of m for $n = 99$ (*left panel*) and $n = 100$ (*right panel*) for quasi-equipopulous units (*crosses*). The *solid lines* denote the real-extensions from Sect. 2

Fig. 5 The MMD as a function of m for $n = 20$ for quasi-equipopulous units. *Crosses*: first method; *circles*: second method. Note that the methods agree (at least to a good approximation) almost everywhere



Here the second equation is used to define the real numbers $c_j(n_j)$. As Beisbart and Bovens (2007) show in their Theorem 1, sensitivity is maximized under the following rule in the board: Each unit is assigned a weight proportional to $c_j(n_j)$ and there is acceptance iff the sum of the weights of the Yes-votes exceeds half of the sum of all weights. Accordingly, in our second method, we will assign each unit a weight that is proportional to $c_j(n_j)$.

Results for $n = 20$ can be seen in Fig. 5.[6] We observe that there are almost no differences between our second and our first method. The reason is as follows. Note first that the new weights $c_j(n_j)$ are not far from equal weights because the units are almost equal-sized. Further, often slight changes in voting weights do not change the voting power. As an example, consider $n = 20$ and $m = 3$. If there is equal representation of the units in the board, i.e., if each representative has the same weight, a representative is pivotal iff the other two representatives cast different votes. Consider now the case in which each representative j is given a weight proportional to $c_j(n_j)$. In our case, each $c_j(n_j)$-value will roughly be proportional to $\sqrt{n_j}$.[7] That is, the $c_j(n_j)$-values will be proportional to $\sqrt{7}$ (twice) and to $\sqrt{6}$ (once),

[6] For obtaining these results, we used a Mathematica package "Banzhaf" available under http://library.wolfram.com/infocenter/MathSource/3592 (checked May 2011).

[7] We are here using an approximation according to which the probability of pivotality in simple majority voting with k voters is proportional to the inverse square root of k (Felsenthal and Machover 1998: 56).

respectively. As a consequence, a representative is still pivotal iff the other representatives cast different votes. Thus, the probability of pivotality is not affected, as we switch to our second method, and neither is sensitivity.

Similar results as those that we have obtained here are also to be expected for larger values of n, at least if m is smaller than $\sqrt{n}$. We will therefore not consider our second method any further.

Altogether, we draw the following conclusions for quasi-equipopulous units. Independently of whether n is even or odd, we obtain curves with several local maxima. Local maxima arise if m or n/m are even. The worst possible case—the case in which the MMD has a global maximum—seems to be around $m = 2$ and $m = n/2$.

5 Larger deviations from equipopulous units

Quasi-equipopulous units will not be an option in many applications.[8] For instance, in the United States it would be extremely difficult to ensure quasi-equipopulous districts. Congressional districts or similar units will always follow geographical boundaries. Consequently, as people move, the population sizes change. Districts may be reshaped from time to time, but it is completely unrealistic to reshape districts following the moves of single families. This raises the question how the MMD behaves as a function of the number of units if larger deviations from equipopulous units are allowed.

Inspired by the example of the United States, we delimit ourselves to very large populations and fairly small numbers of units. We furthermore assume that there is equal representation in the board of representatives. That is, although the sizes of the units may differ significantly, each representative has the same weight. This restriction may be justified by appealing to practical concerns. It may simply be too difficult to monitor the sizes of the units and to adjust the weights accordingly.

We start again from (1) and (8). However, to simplify things, we will consistently approximate the probability that a voter (be it a citizen or a representative) is pivotal under simple majority voting with k voters by $\sqrt{2/\pi k}$, unless k is smaller than 1,000 (in which case no approximation for the probability of pivotality is used).

When we allow for larger deviations from equipopulous units, an m-value does not uniquely fix the partition any more. We will therefore average over many partitions for each value of m. Assume now that m is fixed at some integer value. Under each partition into m units, unit j has a population of $n_j = (1 + \varepsilon_j) \times n/m$, where ε_j is a random variable. It quantifies the relative deviation from the average size of the units. Of course, summing over the ε_j must produce zero. Different random numbers yield different realizations of a partition into m units. For each random variable ε_j, we assume equiprobability in the interval $[-0.2, 0.2]$ and draw random numbers independently. We then renormalize the populations to make sure that the sum constraint is fulfilled. Of course, in this way, the sizes of the units will mostly not be integer-valued. This is not a problem, though, as the units are always assumed to be large, in which case the probability of pivotality can well be approximated using by $\sqrt{2/\pi k}$ for k voters. This yields a good real-extension, which does not much vary as a function of the number of voters k.

In Fig. 6, we assume that $n = 100{,}000{,}000$ (this number provides a rough approximation of the number of voters in the United States) and show the average MMD. Each average arises from thousand random realizations of a partition with m units. We focus on the most interesting case of a relatively small number m of units.

[8] We are here following a suggestion made by a referee to whom we are grateful.

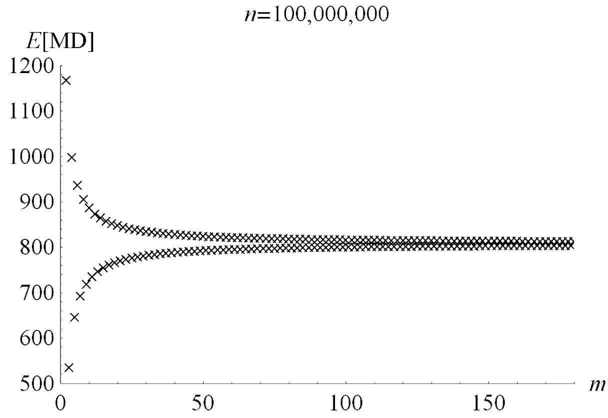


Fig. 6 The average MMD for larger deviations from equipopulous units

In the figure we see two branches, which rapidly approach a constant function. The reason is as follows If m is larger than, say 100, but smaller than $n/100 = 1{,}000{,}000$, both the probability that a voter is pivotal in her unit and the probability that her representative is pivotal in the board can well be approximated by the inverse square root of the size of the respective electorate. In this limit, the sensitivity for a partition is

$$S = \sum_{j=1}^{m} \left(\frac{n}{m}(1+\varepsilon_j) \right) \times \sqrt{\frac{2}{\pi m}} \times \sqrt{\frac{2}{\pi n (1+\varepsilon_j)/m}}.$$

If we assume that each ε_j is zero, we obtain an MMD that does not depend on m any more:

$$E[\mathrm{MD}] = 0.5 \times \left(S_n - m \times \frac{n}{m} \times \sqrt{\frac{2}{\pi m}} \times \sqrt{\frac{2m}{\pi n}} \right) = 0.5 \times \sqrt{n} \times \left(\sqrt{\frac{2}{\pi}} - \frac{2}{\pi} \right).$$

If the ε_j-values are allowed to differ from zero, the average MMD is a bit larger, but there is still no m-dependence, or only a very weak one.

The two branches for small values of m approach the (almost) straight line for large m-values from above and below, respectively. The upper branch corresponds to even values of m, the lower to odd values of m. Given our earlier results, it should not be surprising that odd values of m tend to be better than even values.

In Fig. 6, we show only the average values of the MMD. That is, each point arises by averaging over 100 realizations of the random numbers. We have also quantified the fluctuations; they are very small. In the graph, the size of the error bar would not often exceed the size of the symbol. We obtain relatively large fluctuations for small values of m. The root mean square fluctuations are never larger than 4.

Our result yields a clear recommendation for a partition into a number of units that is odd and the smaller this number, the better.

6 Discussion

To discuss the political relevance of our findings, let us first provide a brief summary of our results.

We have investigated how the mean majority deficit behaves for a two-tiered voting system when the population of n voters is split into different numbers of more or less equipopulous constitutive units. We have assumed binary votes and the Bernoulli probability model. Within each unit, there is a simple majority vote. The votes of the units are amalgamated using simple majority voting or a weighted voting rule that is in a certain sense optimal (see Sect. 4). Clearly, the mean majority deficit is minimal (viz. zero) if there is one unit or if there are as many units as there are people. Our focus is thus on numbers of units in between these extremes and our concern is to steer away from the maximal mean majority deficit. If the units are exactly equipopulous, our results depend significantly on whether the size of the total population is an odd or an even number (Sect. 3). For odd population sizes, we obtain a new square-root rule. The mean majority deficit can naturally be extrapolated using a function that has its maximum at the square root of the total population size. For even population sizes, two units or units with two numbers of voters are worst. If the units are not necessarily exactly equipopulous, but close, the mean majority deficit is also maximal for partitions with two units or with a large number of units that have two voters each (Sect. 4). Finally, if we allow for larger fluctuations in the population sizes, small even numbers of units continue to be bad for minimizing the mean majority deficit (Sect. 5).

Our results are clearly relevant for the institutional design in companies, organizations or federations. Proposals are voted on within each unit and a representative on the board will then cast a vote in accordance with the majority vote within his or her unit. Now if the board decision is out of line with the popular vote then this threatens the legitimacy of democratic decision-making. This is the majoritarian concern (see Felsenthal and Machover 2000: 23–25). So what kinds of partitions are more and less subject to the threat of high MMDs, assuming equipopulous (or close to equipopulous) units? The lesson is roughly that small even-sized units and small even-sized boards tend to be more pernicious, whereas small odd-sized units and small odd-sized boards tend to be more attractive on strictly majoritarian grounds. Note, however, that, for a moderate number of units—i.e., a number of units that is not close to zero but not much greater than the square root of the total population size—the MMD is constrained within very narrow bounds and, hence, there is not too much discrimination possible on majoritarian grounds.

The majoritarian concern is one amongst others in designing a partition. There is also a *motivational* concern. Voters feel more engaged in the process of decision-making when they can vote within a unit with which they feel a sense of allegiance and when there is a public record of these votes. In a one-tier or population-wide vote they may feel alienated from the process and the turnout may be low. Furthermore there is a *deliberational* concern. Voting in a board is often informed by deliberation in that very board and we may want to secure that the groups are neither too small nor too large to enable meaningful deliberation. Motivational and deliberational concerns need to be balanced against the majoritarian concern in designing two-tiered voting systems.

Let us now consider a special application, viz. the US Electoral College (EC). Opposition against the EC tends to flare up after election in which a president was elected with a majority deficit. This was the case in the 2000 election when Bush was elected, while Gore received more votes than Bush. There is some sentiment that the winner-takes-all is to blame for majority deficits. In response, we have witnessed the *Presidential Electors Initiative* in California in which each elector would cast a vote in accordance with the majority vote in her district and two electors would cast a vote in accordance with the majority vote in the state. In *Why the Electoral College is Bad for America*, Edwards (2004: 36–38) argues against the winner-takes-all system because it does not distinguish between large and small majorities and this is what causes majority deficits. This is easy to see—if Bush gets the majority of EC votes from states in which he has a small majority and Gore gets a minority of EC votes from states in which he has a large majority then it is clear how Gore can lose the election even with a majority of the popular vote. Edwards then provides a list of early statesmen who all supported some version of the district plan under which smaller, but roughly equipopulous districts each have one representative.[9] But no such plan became law. The more powerful state parties in each state supported the winner-takes-all system ‘precisely because it distorted the popular will and allowed them to reap the benefits of the state’s electoral votes (Edwards 2004: 38)’. Now the reasoning that Edwards attributes to the state parties presupposes that a district plan is more respectful of the popular will. One interpretation is that a vote is somehow closer to the popular vote if majorities are registered at the level of the district—i.e. closer to the people—rather than at the level of the state. But does this argument hold? Is it the case that subdividing the unit at which the popular vote is registered is more respectful of the popular vote?

If it is generally true that subdividing units yields results that are typically more respectful of the popular vote, this should also hold under the assumption of equipopulous states. The argument would then not be that we are reducing the MMD by moving from unequal-sized states to equal-sized districts. The argument would rather be that we would reduce the MMD by moving closer to the people when we move from larger states to smaller districts. If ‘getting closer to the people’ by moving from states to districts is meant to lower the MMD then it should do so for unequal-sized as well as for equal-sized states.

Our results show that this is not so. In the US Electoral College, the winner-takes-all system operates with 50 states and the District of Columbia, whereas a system with congressional districts operates with 435 districts. *Under the assumption of equal-sized units*, the mean majority deficit is smaller for 51 states than it is for 435 districts. To show this we use the method from Sect. 5. That is, we assume a total voter turnout of $n = 122{,}295{,}345$, as we witnessed in the US presidential elections in 2004,[10] and split the United States into 51 or 435 units. In each case, we allow for random fluctuations of the population sizes within the units and calculate the average MMD. We obtain an average $E[\text{MD}] = 880$ for 51 states, and an average $E[\text{MD}] = 896$ for 435 districts.[11] We obtain similar results if we use the second method from Sect. 4: The MMD increases by about 16 when we move from 51 to 435 units. Thus, splitting up the federation into reasonably-sized smaller units to avoid the misgivings about winner-takes-all in large states may have certain advantages (e.g., it may address motivational concerns), but it worsens the expectation that votes will be marred by majority deficits and hence should not be defended as a safeguard for the legitimacy of democratic institutions in this respect.

Admittedly, the increase in the MMD is small, as we move from a partition into 51 states to a partition into 435 districts. Nevertheless, our results show clearly that it is wrong to expect that the MMD generally goes down, as we increase the number of units in this range of m-values.

But what if we start from $(50 + 1)$ unequal-sized states and would move to a system with roughly equal-sized districts? This reflects the political reality in the United States. How

[9] See Miller (2009, Sect. 5) for varieties of such plans.

[10] Voter turnout taken from www.fec.gov/pubrec/fe2004/tables.pdf.

[11] Note that this is not the pure district plan (Miller 2009: 360–361) under which each state has as many equal-sized districts as it has electors, adding up to 538 districts. But if we carry out our calculations with 538 equal-sized districts with fluctuations in the population (for which we obtain an MMD of 899) we should obtain a good approximation of the MMD for the pure district plan. We do not consider the modified district plan under which each congressional district has an elector whose vote registers the majority vote in the district and each state has two electors whose votes register the majority vote in the state (*ibid.*).

does this affect the mean majority deficit? If we assume that the electors of each state are determined by a winner-takes-all rule (which in reality they are not in Maine and Nebraska), we obtain a MMD of about 1057.[12] Thus, the present system is worse than either a partition into 51 equal-sized states or a partition into 435 equal-sized districts. The reason seems to be that large fluctuations in the numbers of people per unit tend to increase the MMD (note that the fluctuations in the population sizes of the US states are much larger than those considered in Sect. 5). We have not been able to derive any analytical generalizations about shifts from unequal-sized units to equal-sized subunits so far, but we hope to come back to this question in our future research.

In conclusion, a move from the present system to a district system would reduce the MMD. But this is not because recording votes at the district level rather than at the state level is somehow closer to the people. That reasoning is fallacious. District-based systems are advantageous compared to the present system not because the constitutive units in the district-based system are smaller but rather because the constitutive units in the present system, i.e., the states are of highly unequal sizes.

Acknowledgements This paper goes back to a conversation between one of us (L.B.) and Nicholas R. Miller, to whom our first thanks go. An earlier version of this paper was presented at the Voting Power in Practice Workshop at the University of Warwick in July 2009, sponsored by the Leverhulme Trust (Grant F/07-004/AJ). We thank the organizers and the participants for interesting feedback. Finally, we are grateful to William F. Shughart II and two anonymous referees for helpful suggestions.

Appendix 1: Further analytical arguments concerning equipopulous units

In this appendix we consider exactly equipopulous units and assume that the size of the population n is odd. Our aim is to strengthen our results concerning the maximization of $E[\mathrm{MD}]$. Our starting point is provided by (1) and (9). Above we have already shown that the first derivative of $E[\mathrm{MD}]$ is zero at $m = \sqrt{n}$. But we have not yet shown that $E[\mathrm{MD}]$ reaches its maximum at this point. For further analytical argument, we consider approximations of $E[\mathrm{MD}]$. $E[\mathrm{MD}]$ contains binomials and, hence, factorials. Accordingly, the real-extended $E[\mathrm{MD}]$ contains Gamma functions. As is well known, a Gamma function may be approximated by what is sometimes called the Stirling series. The Stirling series starts with (see Morse and Feshbach 1953: 443):

$$k! = \Gamma(k+1) \approx \sqrt{2\pi k}e^{k\ln(k)-k}\left(1 + \frac{1}{12k} + \cdots\right).$$

However, this series does not converge (see, e.g., Havil 2003: 86–88). Care is therefore required in using this series for approximations.

We consider tentatively a few approximations by keeping the first r addends in the bracket of the Stirling series. We consider $r = 1$ and $r = 2$. For $r = 1$, we obtain Stirling's formula:

$$k! \approx \sqrt{2\pi k}e^{k\ln(k)-k}. \tag{a.1}$$

Stirling's formula is commonly used to approximate a binomial distribution by a Gaussian.

[12]Here we have again assumed total voter turnout of $n = 122{,}295{,}345$. These votes are distributed to the states using VAP data of 2003 taken from http://www.census.gov/Press-Release/www/releases/CB04-36TABLE1.pdf. For the calculation we used the program under http://www.math.temple.edu/cow/bpi.html. Cf. Beisbart and Bovens (2008).

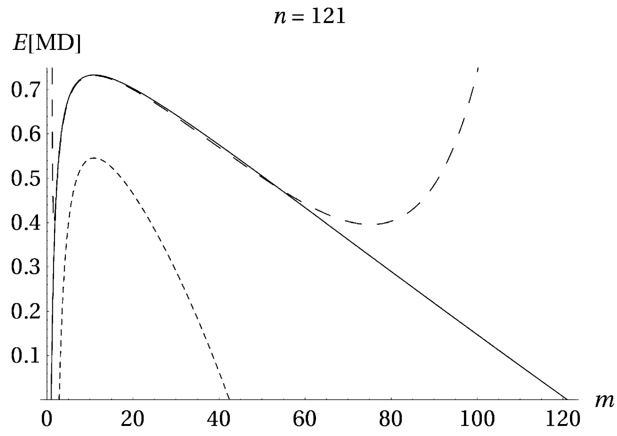


Fig. 7 The exact result (*solid line*), the approximation A1 (*short-dashed*) and A2 (*long-dashed*)

For $r = 2$, we obtain what we call the improved Stirling's formula

$$k! \approx \sqrt{2\pi k} e^{k \ln(k) - k} \left(1 + \frac{1}{12k}\right). \tag{a.2}$$

Equation (1) for the mean majority deficit contains factorials of $(m-1)$, $(m-1)/2$, $(n/m - 1)$ and $(n/m-1)/2$ because of (8). We obtain two approximations for the mean majority deficit by consistently replacing the factorials by the approximations (a.1) and (a.2), respectively; call them (A1) and (A2). Both approximations are only good for $m \gg 1$ and $n/m \gg 1$, which is the middle range of our graphs.

The results for both approximations for $n = 121$ are shown in Fig. 7, where the short-dashed line represents the approximation for $r = 1$, whereas the long-dashed line represents the approximation for $r = 2$. Note first that both curves have a maximum at $m = \sqrt{n}$. It can in fact be shown that every approximation from the Stirling series has a zero derivative at this place; the reason is that each Stirling series approximation for the probability of double pivotality can be written in the form (F) as defined after (9).

We can see in Fig. 7 that the first approximation is not close to the analytic results, though it reaches its maximum for the same value of m, whereas the second approximation is close to the analytic results for a sufficiently broad range of intermediate m-values.

Both approximations help us to constrain the shape of the curve. The derivative of A1 with respect to m reads

$$5n \frac{n - m^2}{\pi \sqrt{m}(n-m)^{3/2}(m-1)^{3/2}}.$$

In the interval $(1, n)$, this derivative equals zero for exactly one value of m, viz. for $m = \sqrt{n}$, and we can see immediately that the derivative changes its sign from positive to negative, as m increases and crosses $\sqrt{n}$. As a consequence, A1 predicts exactly one maximum of the mean majority deficit in $(1, n)$, with a location at $m = \sqrt{n}$. This is the result we were trying to establish. Still, arguing from A1 alone is problematic, since A1 does not match the analytic curve very well.

A2 is more difficult to investigate. In the interval $(1, n)$, its derivative has three zeroes, one of them at $m = \sqrt{n}$. One can prove that, for $n \geq 6$, the second derivative at $m = \sqrt{n}$ is negative such that we have a maximum. It follows that, for $n \geq 6$, the other zeroes of the derivative cannot be local maxima.

To sum up, our analytical arguments from Sect. 2 have already shown that the real-extended $E[\text{MD}]$ has a zero derivative at $m=\sqrt{n}$. The approximations indicate that this point is a maximum.

Appendix 2: Lotteries of voting games

We consider a two-tier voting system of the following type. The vote of each unit (or of its representative) is fixed in the following way: If more than one half of the votes in the unit are Yes (No), the vote of the unit is Yes (No). If there is a tie in the unit, a fair coin is flipped. Likewise, the final outcome of the vote is Yes (No) if more than one half of the representatives vote Yes (No). If there is a draw in the board of representatives, a fair coin is flipped. Since there cannot be draws in votes with odd numbers of votes, the coin-flipping mechanism is implemented only in a unit or in the board of representatives if its number of votes is even.

Consider now the vote in some unit or in the board of representatives. Suppose that there are k votes, where k is even. Now effectively what the coin flip does in the case of a tie is to pick the voting rule or voting game that is operative for this particular vote. If the flip effects a Yes-vote, then effectively the quota is set at $k/2$ (i.e., $k/2$ or more votes are needed for a Yes outcome) and the voting game $\text{M}_{k,k/2}$ is operative in the terms of Felsenthal and Machover (1998: 26). If the flip effects a No-vote, then effectively the quota is set at $(k/2+1)$ and the voting game $\text{M}_{k,(k/2+1)}$ is operative. Since the quotas $k/2$ and $(k/2+1)$ do not make a difference when there is no tie, we may also describe the vote in our unit/board as follows: First a coin is flipped that fixes the threshold of acceptance for this particular vote; second, a vote is taken following a voting rule with this very quota. In this sense we may speak of a lottery over voting games (see Laruelle and Valenciano 2004).

Our two-tier voting system in which each simple majority vote with an even number of votes has been replaced by a lottery can now be thought of as one big lottery of voting games. In a first step, a number of coins are flipped in order to fix the quotas at various places in the two-tier voting system. Second, a vote is taken following the two-tiered voting rules that arise. The MMD of this big lottery of voting games can now be calculated by adding up the MMDs for each of the voting systems that may result from flipping the coins with the appropriate probabilistic weights. We will now show that each voting system that may result from flipping the coins has the same MMD as our original voting game. It follows that the results for the MMD are not different from our results that we have obtained in Sect. 3.1.

For each voting system that may result from flipping the coins, we can still apply (1) and add up the powers of the citizens' votes for obtaining sensitivity, and the voting power for each voter is still the product of $P(\textit{citizen} \to \textit{unit})$ and $P(\textit{unit} \to \textit{br})$. The voting systems that may result from the coin flips differ only in that sometimes other thresholds are selected in some units and/or the board of representatives. It turns out that the values of $P(\textit{citizen} \to \textit{unit})$ and $P(\textit{unit} \to \textit{br})$ both are the same independently of what the thresholds are in the case of an even number of votes at one or both levels. The reason is that the chances of a vote being pivotal are equal under $\text{M}_{k,k/2}$ and under $\text{M}_{k,(k/2+1)}$. Following (3), they are $\binom{k-1}{k/2-1}/2^{k-1}$ and $\binom{k-1}{k/2}/2^{k-1}$, which are equivalent.

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